Vertex Guarding in Weak Visibility Polygons

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(Joint work with Subir Kumar Ghosh and Bodhayan Roy)

Polygons and Visibility

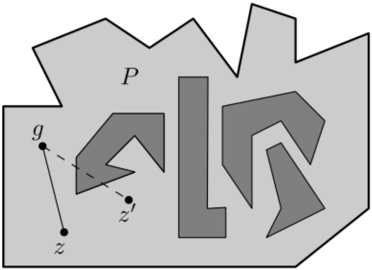

Figure: Polygon with holes

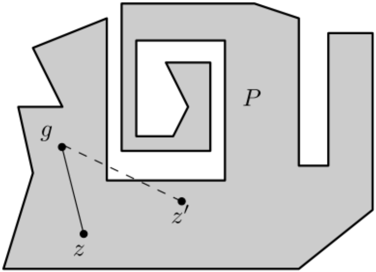

Figure: Polygon without holes

Definition (Visibility of a Point)

Any point $z \in P$ is said to be *visible* from another point $g \in P$ if the line segment zg does not intersect the exterior of P.

Art Gallery Problem

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Victor Klee (1973) $\rightarrow$ How many point guards or vertex guards are always sufficient to guard a simple polygon having n vertices?

Sufficient Number of Guards

Theorem (Chvatal (1975), Fisk (1978))

For guarding a simple polygon with n vertices, $\lfloor \frac{n}{3} \rfloor$ point guards or vertex guards are sufficient and sometimes necessary.

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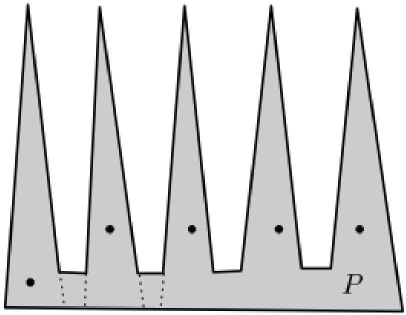

Figure: A polygon where $\lfloor \frac{n}{3} \rfloor$ point guards or vertex guards are necessary.

Art Gallery Problem - Hardness Results

Definition (Decision Version of the Art Gallery Problem)

Given a polygon P and a number k as input, can the polygon P be guarded with k or fewer guards?

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- Proved to be NP-complete for point guards (Aggarwal).
- Proved to be APX-complete (Eidenbenz, Stamm and Widmayer), implying that no PTAS can exist for AGP.
- Specifically for polygons with holes, AGP cannot be approximated to within a factor of $\Omega(\ln n)$ (Eidenbenz, Stamm and Widmayer).

Art Gallery Problem - Approximation Algorithms

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Conjecture (Ghosh (1987))

There exist polynomial time algorithms with a constant approximation ratio for vertex guarding polygons without holes.

Our Contributions

Definition (Weak Visibility Polygon)

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We present:

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A polygon P is said to be a *weak visibility polygon* if every point in P is visible from some point of an edge uv.

We present:

1. A 6-approximation algorithm, which has running time $\mathcal{O}(n^2)$, for vertex guarding polygons that are weakly visible from an edge and contain no holes.
2. A reduction from Set Cover to show that, for the special class of polygons containing holes that are weakly visible from an edge, there cannot exist a polynomial time algorithm for the vertex guard problem with an approximation ratio better than $((1-\epsilon)/12)\ln n$ for any $\epsilon > 0$, unless NP $=$ P.

Euclidean Shortest Path Tree

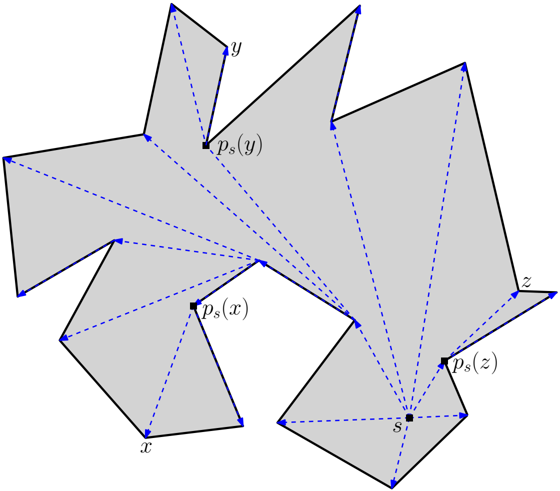
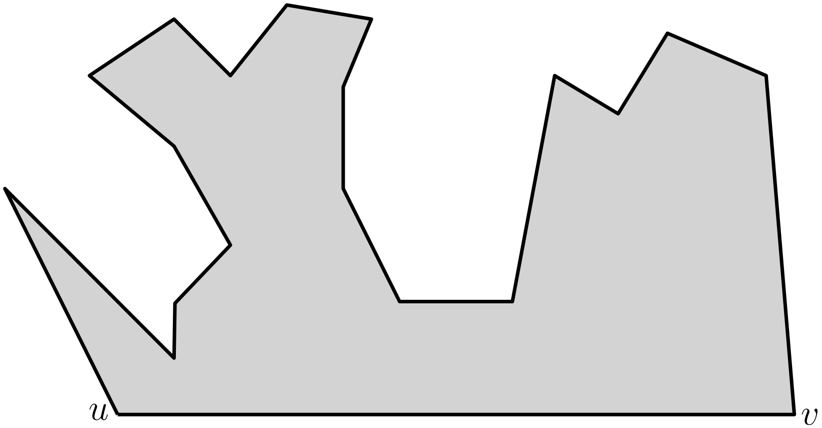


Figure: Euclidean shortest path tree rooted at s. The parents of vertices x, y and z in $SPT(s)$ are marked as $p_s(x)$, $p_s(y)$ and $p_s(z)$ respectively.

A Naive Algorithm for Guarding All Vertices



$$A = \{\}; S_A = \{\}$$

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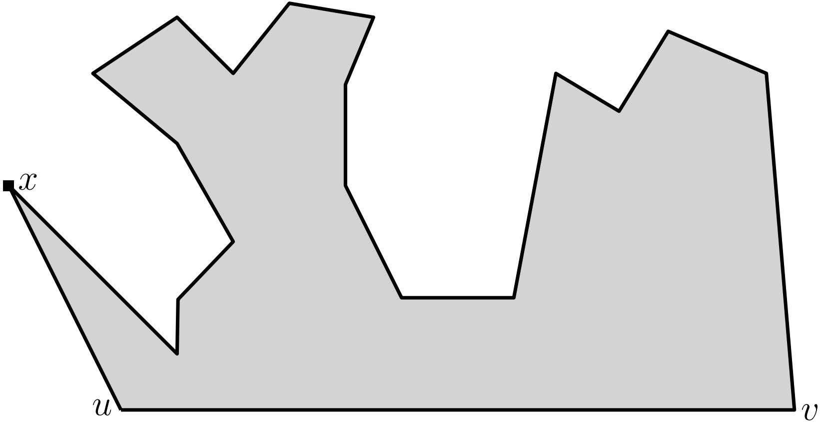

$$A = \{x\} \; ; \; S_A = \{\}$$

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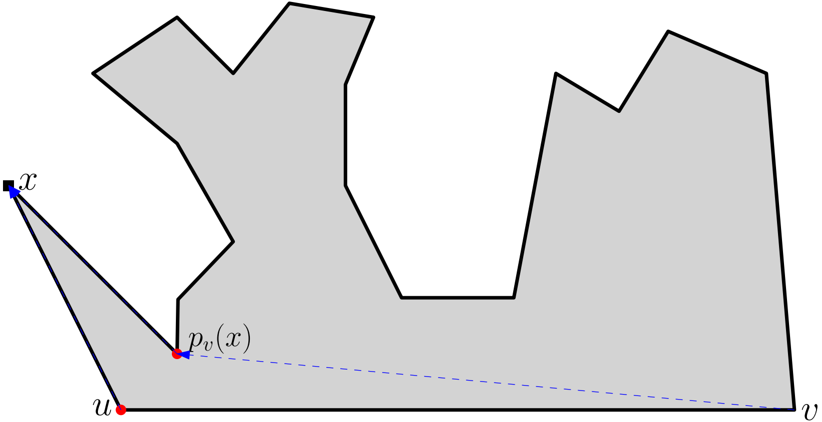

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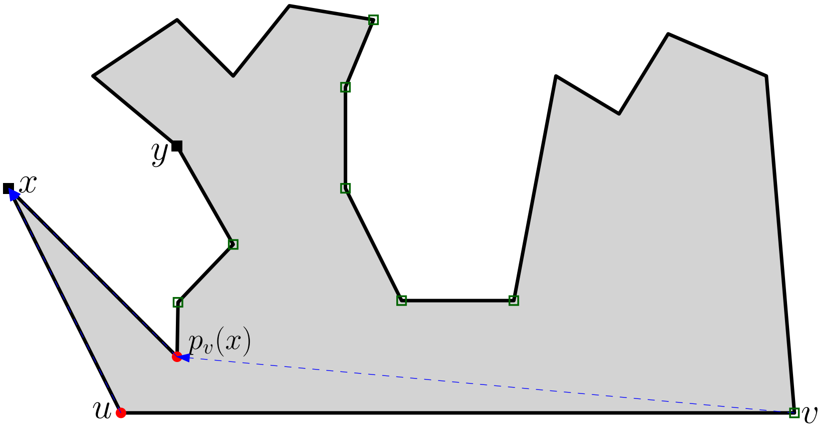

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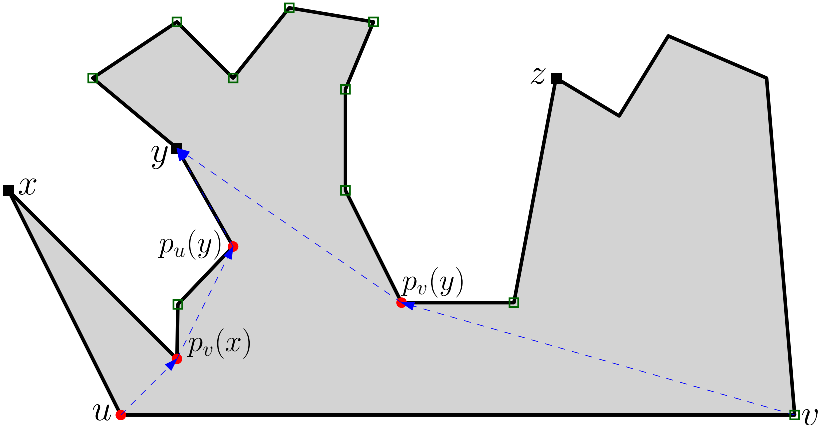

$$A = \{x, y\}\ ;\ S_A = \{u, p_v(x), p_u(y), p_v(y)\}$$

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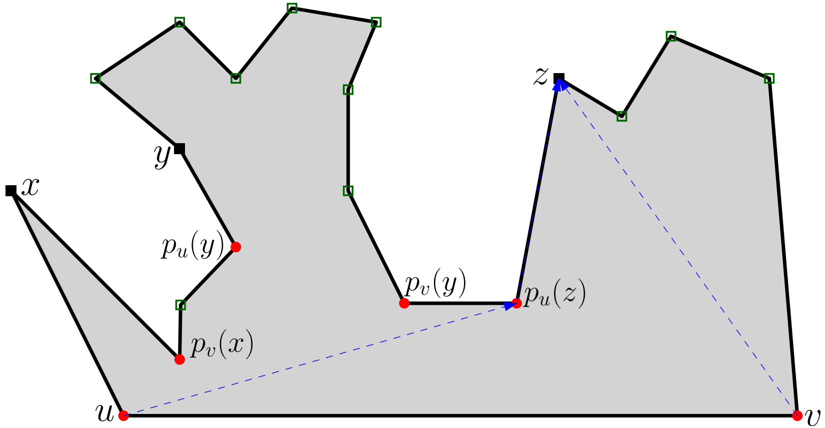
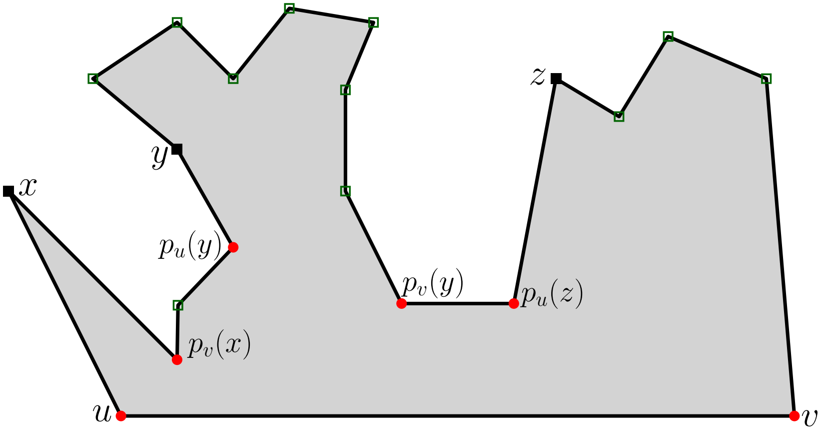


$$A = \{x, y, z\} \; ; \; S_A = \{u, p_v(x), p_u(y), p_v(y), p_u(z), v\}$$

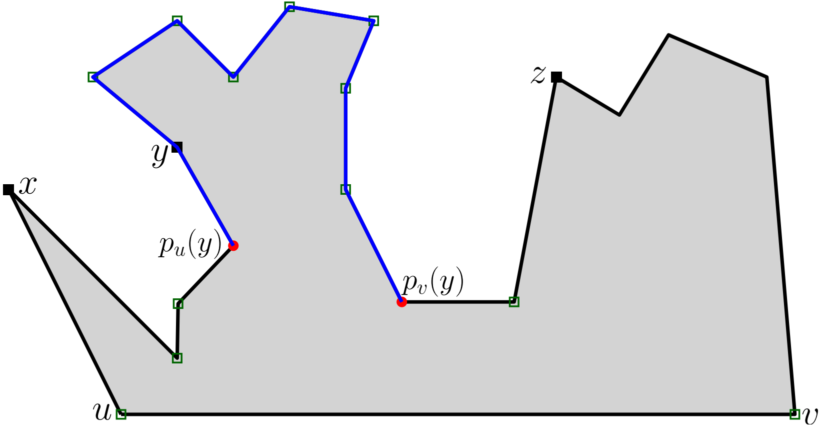
A Naive Algorithm for Guarding All Vertices



$$A = \{x, y, z\} \; ; \; S_A = \{u, p_v(x), p_u(y), p_v(y), p_u(z), v\}$$

N.B. - $|S_A| = 2|A|$

Performance Guarantee under a Special Condition



N.B. - The vertex $y \in A$ is such that every vertex lying on the clockwise boundary between $p_u(y)$ and $p_v(y)$ (henceforth denoted as $bd_c(p_u(y), p_v(y))$) is visible from $p_u(y)$ or $p_v(y)$.

Performance Guarantee under a Special Condition

Lemma

If each vertex $z \in A$ *is such that every vertex of* $bd_c(p_u(z), p_v(z))$ *is visible from* $p_u(z)$ *or* $p_v(z)$*, then* $|S_A| \leq 2|S_{opt}|$.

Performance Guarantee under a Special Condition

Lemma

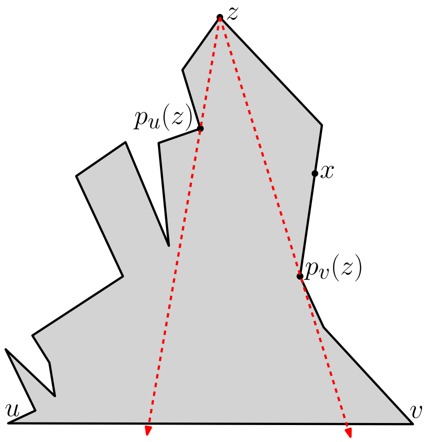
If each vertex $z \in A$ is such that every vertex of $bd_c(p_u(z), p_v(z))$ is visible from $p_u(z)$ or $p_v(z)$, then $|S_A| \leq 2|S_{opt}|$.

Proof.

- $|S_A| = 2|A|$
- $|A| \leq |S_{opt}|$ (to be shown next)
- Therefore, $|S_A| = 2|A| \leq 2|S_{opt}|$

□

Location of an Optimal Guard for Vertex z



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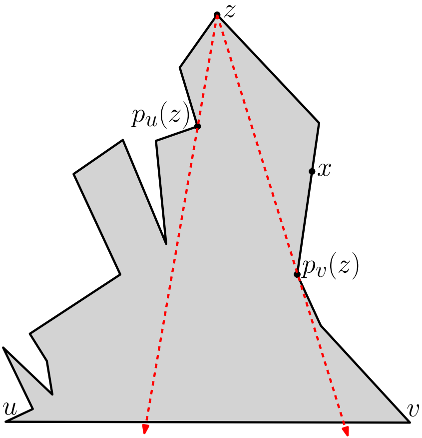
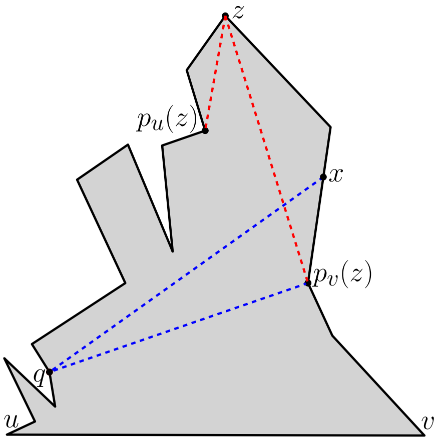


Lemma

Any guard $x \in S_{opt}$ that sees z must lie on $bd_c(p_u(z), p_v(z))$.

Proof sketch of $|A| \leq |S_{opt}|$



- All vertices of $bd_c(p_u(z), p_v(z))$ are visible from $p_u(z)$ or $p_v(z)$.

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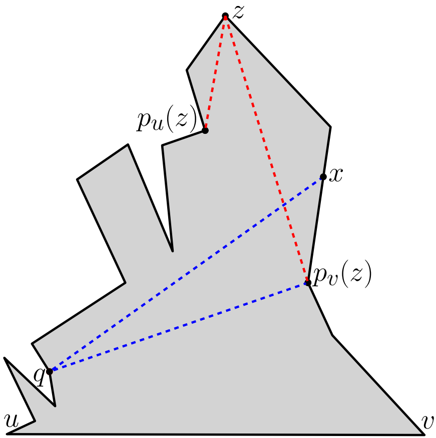

- All vertices of $bd_c(p_u(z), p_v(z))$ are visible from $p_u(z)$ or $p_v(z)$.
- If q is visible from x, then q must be visible from $p_u(z)$ or $p_v(z)$.

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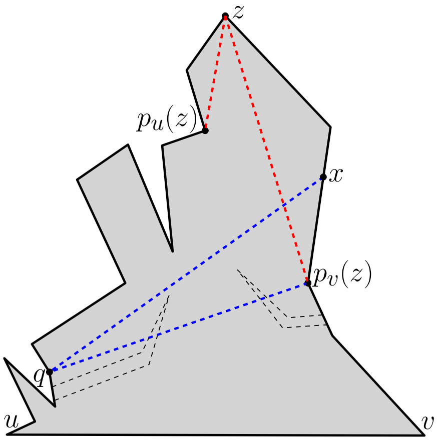

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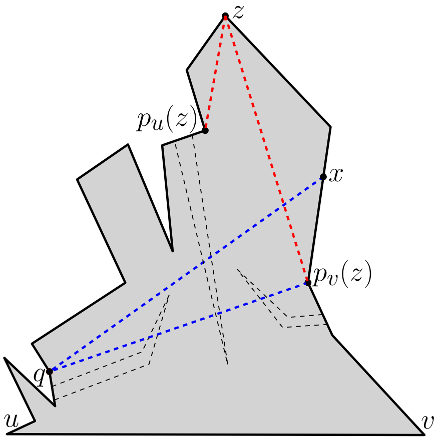

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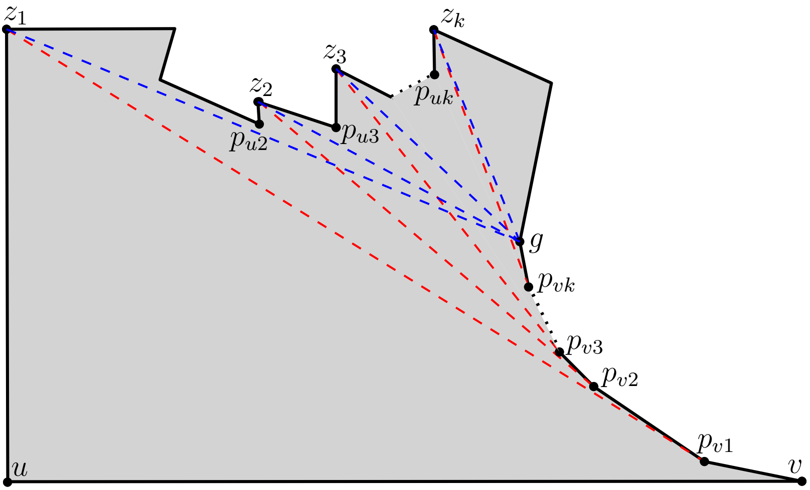
A Bad Input Polygon for the Naive Algorithm



For this input instance, $|S_A| = 2k$, whereas $S_{opt} = \{u, g\}$.

A Better Strategy for Guarding All Vertices

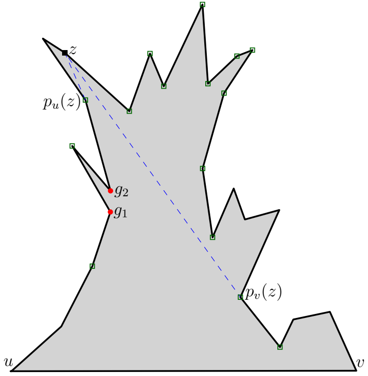
$$B = \{\} \; ; \; S = \{\}$$

New Strategy - Skip some unmarked vertices along the clockwise scan and choose vertices to include in B more carefully!

Invariance - If z is the current vertex under consideration along the clockwise scan, then every vertex of $bd_c(u, z)$ is visible from some guard in $S \cup \{p_u(z), p_v(z)\}$.

A Better Strategy for Guarding All Vertices

Case 1 - Every vertex lying on $bd_c(z, p_v(z))$, except z itself, is either visible already from guards currently in S or becomes visible if new guards are placed at $p_u(z)$ and $p_v(z)$.



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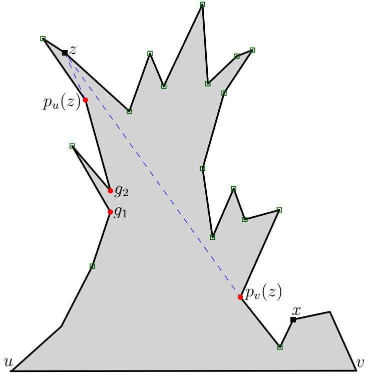
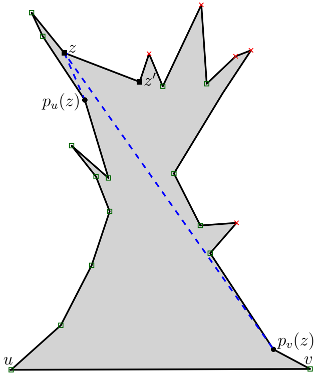


$$B = B \cup \{z\} \; ; \; S = S \cup \{p_u(z), p_v(z)\} \; ; \; z = x$$

A Better Strategy for Guarding All Vertices

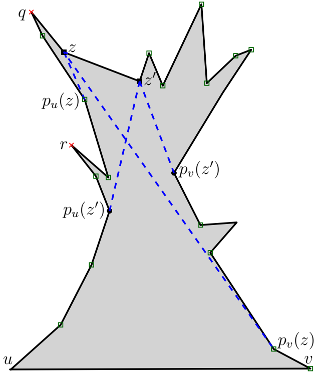
Case 2 - There exist some vertices lying on $bd_c(z, p_v(z))$, not visible already from guards currently in S, such that they do not become visible even if new guards are placed at $p_u(z)$ and $p_v(z)$.



Let z' be the next vertex along the clockwise scan that is not visible from any guard already in S.

A Better Strategy for Guarding All Vertices

Case 2a - Not every unmarked vertex of $bd_c(p_u(z'), z')$ is visible from $p_u(z')$ or $p_v(z')$.



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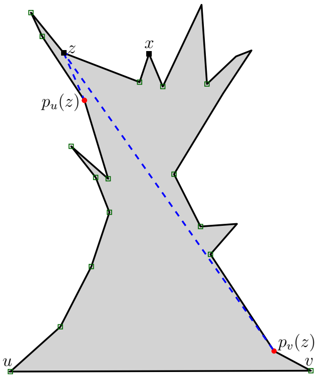

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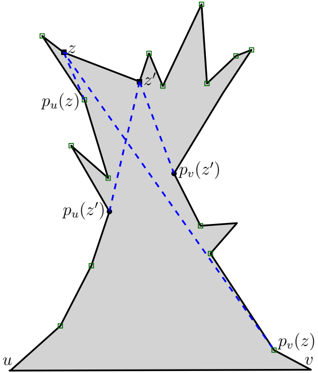
Case 2b - Every unmarked vertex of $bd_c(p_u(z'), z')$ is visible from $p_u(z')$ or $p_v(z')$.



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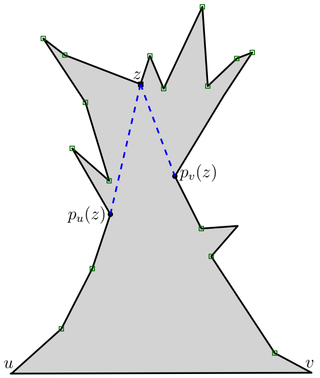
Case 2b - Every unmarked vertex of $bd_c(p_u(z'), z')$ is visible from guards at $p_u(z')$ or $p_v(z')$.



$$B = B \cup \{\} \; ; \; S = S \cup \{\} \; ; \; z = z'$$

Approximation Ratio of our Algorithm

Lemma

$|B| \leq 2|S_{opt}|$.

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Proof.

There exists a bipartite graph $G = (B \cup S_{opt}, E)$ such that:
(a) the degree of each vertex in B is exactly 1, and,
(b) the degree of each vertex in S_{opt} is at most 2. □

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Therefore, $|S| = 2|B| \leq 4|S_{opt}|$. □

Insufficiency of Guards in S to Cover all Interior Points

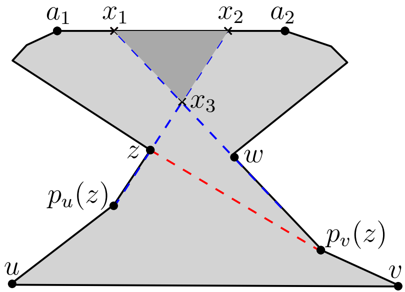

Figure: All vertices are visible from the guard set $S = \{p_u(z), p_v(z)\}$, but all points in the triangular interior region $x_1x_2x_3$ are invisible.

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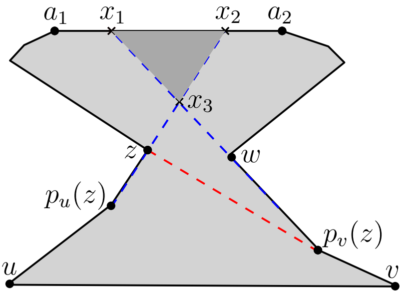

Figure: All vertices are visible from the guard set $S = \{p_u(z), p_v(z)\}$, but all points in the triangular interior region $x_1x_2x_3$ are invisible.

NOTE: One of the sides x_1x_2 of the triangle $x_1x_2x_3$ is a part of the polygonal edge a_1a_2. In fact, for any such invisible region, one of the sides must always be part of a polygonal edge.

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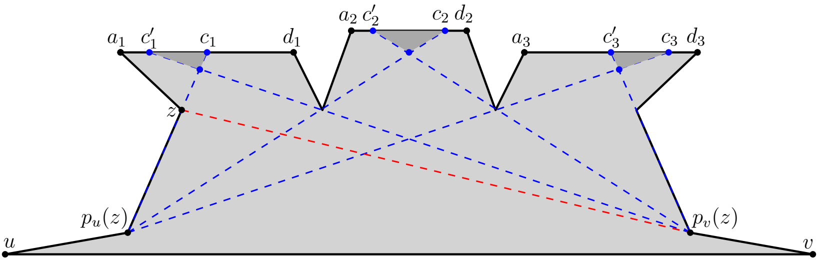

Figure: Multiple invisible regions exist within the polygon that are not visible from the guard set $S = \{p_u(z), p_v(z)\}$.

Placement of More Guards to Cover all Interior Points

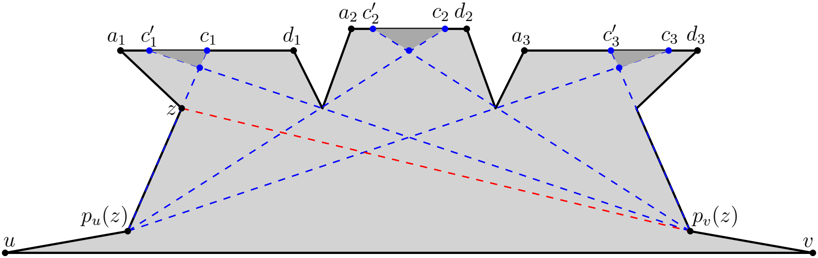

Figure: Multiple invisible regions exist within the polygon that are not visible from the guard set $S = \{p_u(z), p_v(z)\}$.

Lemma

It is possible to choose an additional set of guards S' to cover all invisible regions such that $|S'| \leq 2|S_{opt}|$.

Approximation Ratio of our Algorithm

Theorem

Our algorithm has an approximation ratio of 6.

Proof.

The final guard set returned by our algorithm is $|S \cup S'|$.

$$\begin{aligned} |S \cup S'| &= |S| + |S'| \\ &\leq 4|S_{opt}| + 2|S_{opt}| \\ &= 6|S_{opt}| \end{aligned}$$

$\square$

Running Time of our Algorithm

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For a weak visibility polygon P having n vertices, the running time of our algorithm is $\mathcal{O}(n^2)$.

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- Computation of guard set S takes $\mathcal{O}(n^2)$ time.

Running Time of our Algorithm

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For a weak visibility polygon P having n vertices, the running time of our algorithm is $\mathcal{O}(n^2)$.

Proof.

- Computation of $SPT(u)$ and $SPT(v)$ takes $\mathcal{O}(n)$ time.
- Computation of guard set S takes $\mathcal{O}(n^2)$ time.
- Computation of guard set S' also takes $\mathcal{O}(n^2)$ time.

Running Time of our Algorithm

Theorem

For a weak visibility polygon P having n vertices, the running time of our algorithm is $\mathcal{O}(n^2)$.

Proof.

- Computation of $SPT(u)$ and $SPT(v)$ takes $\mathcal{O}(n)$ time.
- Computation of guard set S takes $\mathcal{O}(n^2)$ time.
- Computation of guard set S' also takes $\mathcal{O}(n^2)$ time.
- Hence, the overall running time of our algorithm is $\mathcal{O}(n^2)$.

$\square$

A Known Inapproximability Result

Theorem (Eidenbenz, Stamm and Widmayer (1998))

For polygons with holes, there cannot exist a polynomial time algorithm for AGP with an approximation ratio better than $((1-\epsilon)/12)\ln n$ *for any* $\epsilon > 0$, *unless* $NP \subseteq TIME(n^{\mathcal{O}(\log\log n)})$.

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The above theorem utilizes the following result by Feige -

Theorem (Feige (1998))

Set Cover cannot be approximated to within a factor of $(1-\epsilon)\ln n$ *for every* $\epsilon > 0$ *unless* $NP \subseteq TIME(n^{\mathcal{O}(\log\log n)})$.

Our Inapproximability Result

A modification of their reduction leads us to the following result -

Theorem

For weak visibility polygons with holes, there cannot exist a polynomial time algorithm for the vertex guarding problem with an approximation ratio better than $((1-\epsilon)/12)\ln n$ *for any* $\epsilon > 0$, *unless* $NP \subseteq TIME(n^{\mathcal{O}(\log\log n)})$.

Our Inapproximability Result

A very recent result by Dinur and Steurer -

Theorem (Dinur and Steurer (2014))

Set Cover cannot be approximated to within a factor of $(1 - \epsilon) \ln n$ *for every* $\epsilon > 0$ *unless* $NP = P$.

Our Inapproximability Result

A very recent result by Dinur and Steurer -

Theorem (Dinur and Steurer (2014))

Set Cover cannot be approximated to within a factor of $(1-\epsilon)\ln n$ *for every* $\epsilon > 0$ *unless* $NP = P$.

With this strengthening of Feige's quasi-NP-hardness, our inapproximability result gets improved to -

Theorem

For weak visibility polygons with holes, there cannot exist a polynomial time algorithm for the vertex guarding problem with an approximation ratio better than $((1-\epsilon)/12)\ln n$ *for any* $\epsilon > 0$, *unless* $NP = P$.

Thank You!