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   - Idealised skipping for search
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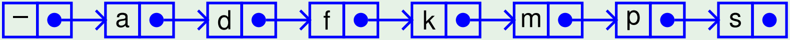
Improving on the linked list

Example (The slow and simple linked list)



Steps to reach “s”: 7

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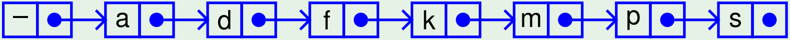

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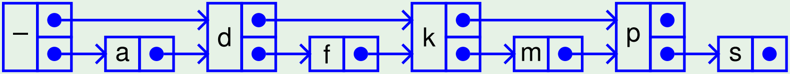

Steps to reach "s": 4

Improving on the linked list

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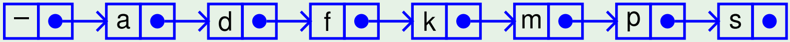

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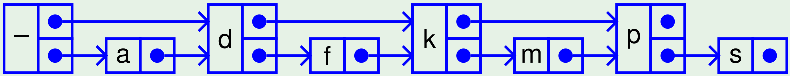

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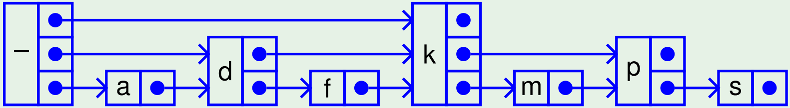
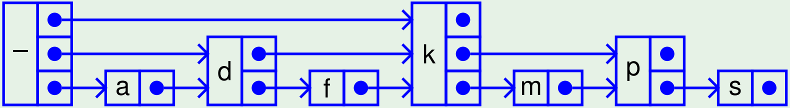


Steps to reach "s": 3

Perfect skip lists

Example (Perfect SL)



- At level 0 all elements are connected as a simple linked list
- All alternate elements at level i are connected as a linked list at level $(i+1)$ – each higher level contains half the elements of the level below it
- Maximum number of levels is $\lceil \lg n \rceil$

Perfect skip lists

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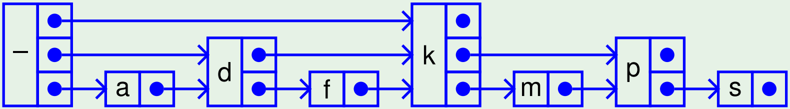
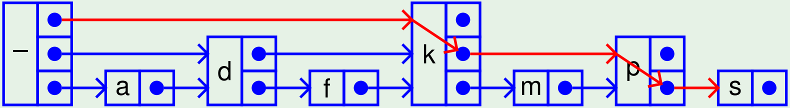


- At level 0 all elements are connected as a simple linked list
- All alternate elements at level i are connected as a linked list at level $(i + 1)$ – each higher level contains half the elements of the level below it
- Maximum number of levels is $\lceil \lg n \rceil$
- Hard to maintain on insertion and deletion, entails $O(n)$ time for insertion and deletion
- Relax the criterion for higher level links from deterministic to probabilistic – proposed by William Pugh around 1989

Searching in a SL

Example (Searching for ‘s’)



Searching for a key ky, starting at the highest level

1. If list is empty, then fail if at the lowest level
   otherwise continue searching after descending one level

Searching in a SL

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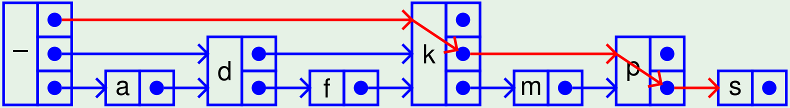

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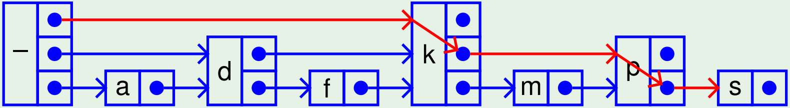

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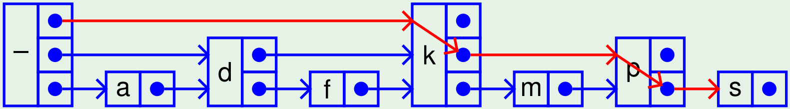

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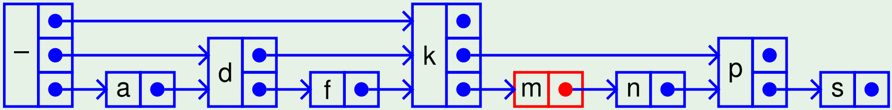
   If yes, then done
3. Stored key at current level $<$ ky ?
   If yes, redo search from one level less, if not at the lowest level
   otherwise fail
4. Now, Stored key at current level $>$ ky
   Continue search from this point

Improving over the perfect SL

Example



- After inserting an element, keep adding levels with a probability of p, which is taken as $\frac{1}{2}$

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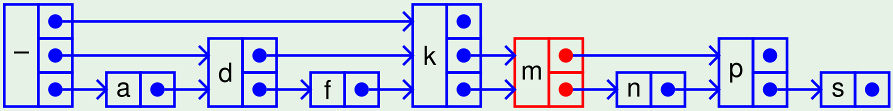

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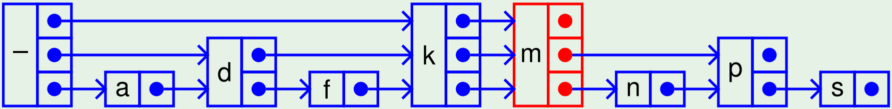

- After inserting an element, keep adding levels with a probability of p, which is taken as $\frac{1}{2}$
- $\Pr[\ell = k] = (1-p)p^k$; $\Pr[\ell \geq k] = p^k$; $\ell \geq 0$
- Note that the number of levels at a key is independent of those of other nodes

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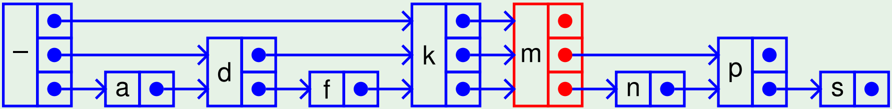

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- $\Pr[\ell = k] = (1-p)p^k$; $\Pr[\ell \geq k] = p^k$; $\ell \geq 0$
- Note that the number of levels at a key is independent of those of other nodes
- Searching is done as explained
- While deleting, a key is simply deleted with all its levels – does not affect the distribution of levels of other nodes!

Distribution of nodes in a (randomised) SL

- Let f_k be the fraction of nodes of an aribtrarily long SL *precisely* at level k (having *precisely* $(k+1)$ forward links)
- Let g_k be the fraction of nodes at level k (or higher) (having $(k+1)$ or more forward links)
- $f_k = (1-p)g_k$, as a node does not get promoted with probability $1-p$
- $g_0 = 1$, $f_0 = (1-p)g_0 = 1-p$
- $g_1 = pg_0$, $f_1 = (1-p)g_1 = p(1-p)g_0 = pf_0$
- $g_k = pg_{k-1}$, $f_k = (1-p)g_k = p(1-p)g_{k-1} = pf_{k-1}$
- $f_0 = 1-p = \frac{1}{2}$ for $p = \frac{1}{2}$
- So, 50% of the nodes are at level 0, if nodes are promoted to the next higher level with a probability of 50%

Expected space usage

- The probability that a node extends to a level k is $\frac{1}{2^k}$, $k \geq 0$
- Let the SL have n nodes
- Expected number of nodes at level k is $\frac{n}{2^k}$
- Let the maximum height of the SL be limited to h
- Expected number of nodes used by the SL is $\sum_{k=0}^{k=h} \frac{n}{2^k} < 2n$
- Thus the space requirement of a SL is $O(n)$

Maximum level in a SL

- Let M be the maximum level of any element in the skip list
- If an element tends to go beyond the maximum level, it will be restricted to M
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- $\mathrm{E}[k > M] = \sum_{k=M+1}^{\infty} n(1-p)p^k = n(1-p)\sum_{k=M+1}^{\infty} p^k = \frac{n(1-p)p^{M+1}}{1-p} = np^{M+1}$
- Let $M = \log_{\frac{1}{p}} n = \log_r n$, where $p = \frac{1}{r}$
- Now, $\mathrm{E}[k > M] = \frac{n}{r^{M+1}} = \frac{n}{r^{\log_r n + \log_r r}} \frac{n}{r^{\log_r nr}} = \frac{1}{r} = p$

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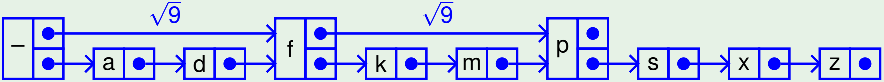
- So, $M = \log_{\frac{1}{p}} n$ is a good choice for the (dynamic) maximum level of the SL

Idealised skipping for search

Example (Baby step, giant step)



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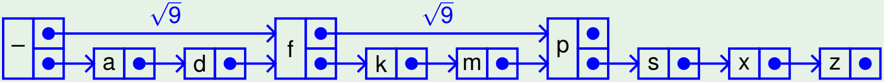

- For a two level skip list structure, it is optimal to have a skip distance of $\sqrt{n}$
  Resulting search time is $2\sqrt{n}$

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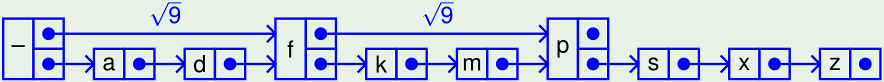

- For a two level skip list structure, it is optimal to have a skip distance of $\sqrt{n}$
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  Search time $3\sqrt[3]{n}$

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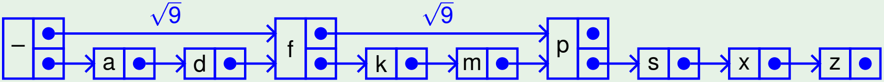

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- For a three level skip list structure, optimal skip distance is $\sqrt[3]{n}$
  Search time $3\sqrt[3]{n}$
- For a k-level skip list structure, optimal skip distance is $\sqrt[k]{n}$
  Search time is $k\sqrt[k]{n}$

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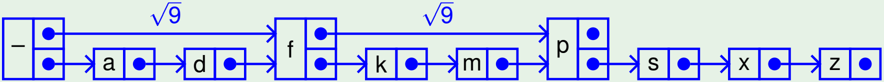

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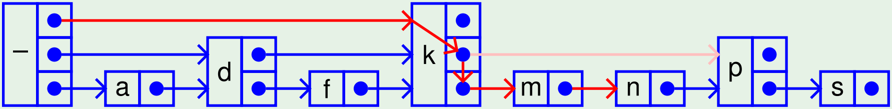
  Search time $3\sqrt[3]{n}$
- For a k-level skip list structure, optimal skip distance is $\sqrt[k]{n}$
  Search time is $k\sqrt[k]{n}$
- If the number of levels is chosen as $\lg n$, the optimal skip distance becomes $n^{\frac{1}{\lg n}} = 2$
  Search time is $(\lg n)\left(n^{\frac{1}{\lg n}}\right) = 2\lg n$

Skipping in the SL during search

Example (Searching for ‘n’ in a SL)



- Necessary to estimate the number of steps to reach an entry
- Will try to workout in the backward direction
- Note that a level is introduced at a node with probablity of $p = \frac{1}{2}$
- Let $S(\jmath)$ denote the number of steps needed to walk back through $\jmath$ levels

Solving for $S(\jmath)$ – steps with $\jmath$-levels remaining

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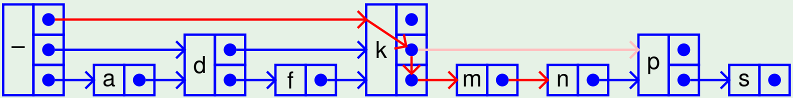

- $S(\jmath) = 1 + pS(\jmath - 1) + (1 - p)S(\jmath)$
- $S(0) = 0$ [already at the top level]
  - 1 for traversing through the current node
  - $pS(\jmath - 1)$ for walking back through $(\jmath - 1)$ levels having ascended a level with probability p
  - $(p - 1)S(\jmath)$ for continuing to walking back through the current level with probability $(1 - p)$

Solving for $S(\jmath)$ – steps with $\jmath$-levels remaining

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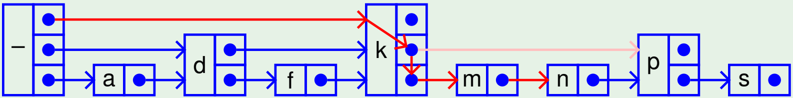

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  - $pS(\jmath - 1)$ for walking back through $(\jmath - 1)$ levels having ascended a level with probability p
  - $(p - 1)S(\jmath)$ for continuing to walking back through the current level with probability $(1 - p)$
- $S(\jmath) = \frac{1}{p} + S(\jmath - 1) = \frac{\jmath}{p}$
- For $p = \frac{1}{2}$, $S(\jmath) = 2\jmath$
- If $M = \log_{\frac{1}{p}} n$ is the maximum level, then $S(M) = 2 \lg n$, for $p = \frac{1}{2}$
- Search time is $O(\lg n)$