Contents

Section outline

1. **Intuitive handling of recurrences**
   - Definition of linear recurrences
   - Simple linear recurrences
   - Departure form linear recurrences

Definition of linear recurrences

Definition (Linear recurrence of order k with constant coefficients)

$$T_n = a_1 T_{n-1} + a_2 T_{n-2} + \ldots + a_k T_{n-k} + g(n) \qquad (1.1)$$

where k is fixed, $a_1, a_2, \ldots, a_k \neq 0$ are constants and $g(n)$ is a real or complex function of n

Definition of linear recurrences

Definition (Linear recurrence of order k with constant coefficients)

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where k is fixed, $a_1, a_2, \ldots, a_k \neq 0$ are constants and $g(n)$ is a real or complex function of n

Definition (Homogenous linear recurrence of order k with constant coefficients)

A linear recurrence with constant coefficients is homogenous when $g(n) = 0$ and has the form

$$T_n = a_1 T_{n-1} + a_2 T_{n-2} + \ldots + a_k T_{n-k} \tag{1.2}$$

Sample recurrences and their solutions

Example (Recurrence for doing one at a time)

$$
\begin{aligned}
T(N) &= 1 && \text{for } N = 1 \\
T(N) &= T(N-1) + 1 && \text{for } N \geq 2
\end{aligned}
$$

Sample recurrences and their solutions

Example (Recurrence for doing one at a time)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = T(N-1) + 1 \qquad \text{for } N \geq 2$$

- Linear but not homogenous of order 1

Sample recurrences and their solutions

Example (Recurrence for doing one at a time)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = T(N-1) + 1 \qquad \text{for } N \geq 2$$

- Linear but not homogenous of order 1
- Apply the method of iteration

Sample recurrences and their solutions

Example (Recurrence for doing one at a time)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = T(N-1) + 1 \qquad \text{for } N \geq 2$$

- Linear but not homogenous of order 1
- Apply the method of iteration
- $T(1) = 1$

Sample recurrences and their solutions

Example (Recurrence for doing one at a time)

$$T(N) = 1 \qquad \text{for } N = 1$$
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- Linear but not homogenous of order 1
- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 1 = 2$

Sample recurrences and their solutions

Example (Recurrence for doing one at a time)

$$
\begin{aligned}
T(N) &= 1 && \text{for } N = 1 \\
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- Linear but not homogenous of order 1
- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 1 = 2$
- $T(3) = T(2) + 1 = 3$

Sample recurrences and their solutions

Example (Recurrence for doing one at a time)

$$T(N) = 1 \quad \text{for } N = 1$$
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- Linear but not homogenous of order 1
- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 1 = 2$
- $T(3) = T(2) + 1 = 3$
- $T(N) = (N-1) + 1 = N \in \mathcal{O}(N)$

Sample recurrences and their solutions

Example (Recurrence for doing one at a time)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = T(N-1) + 1 \qquad \text{for } N \geq 2$$

- Linear but not homogenous of order 1
- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 1 = 2$
- $T(3) = T(2) + 1 = 3$
- $T(N) = (N-1) + 1 = N \in \mathcal{O}(N)$
- Show that this recurrence captures the running time complexity of determining the maximum element, searching in an unsorted array

Sample recurrences and their solutions (contd.)

Example (Recurrence for doing one at a time with full sifting)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = T(N-1) + N \qquad \text{for } N \geq 2$$

Sample recurrences and their solutions (contd.)

Example (Recurrence for doing one at a time with full sifting)

$$
\begin{aligned}
T(N) &= 1 && \text{for } N = 1 \\
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\end{aligned}
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- Linear but not homogenous of order 1

Sample recurrences and their solutions (contd.)

Example (Recurrence for doing one at a time with full sifting)

$$T(N) = 1 \qquad \text{for } N = 1$$
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Sample recurrences and their solutions (contd.)

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- Linear but not homogenous of order 1
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- $T(1) = 1$

Sample recurrences and their solutions (contd.)

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$$T(N) = 1 \qquad \text{for } N = 1$$
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- Linear but not homogenous of order 1
- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 2 = 1 + 2 = 3$

Sample recurrences and their solutions (contd.)

Example (Recurrence for doing one at a time with full sifting)

$$T(N) = 1 \quad \text{for } N = 1$$
$$T(N) = T(N-1) + N \quad \text{for } N \geq 2$$

- Linear but not homogenous of order 1
- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 2 = 1 + 2 = 3$
- $T(3) = T(2) + 3 = 1 + 2 + 3 = 6$

Sample recurrences and their solutions (contd.)

Example (Recurrence for doing one at a time with full sifting)

$$T(N) = 1 \quad \text{for } N = 1$$
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- Linear but not homogenous of order 1
- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 2 = 1 + 2 = 3$
- $T(3) = T(2) + 3 = 1 + 2 + 3 = 6$
- $T(N) = T(N-1) + 1 = 1 + 2 + 3 + \ldots + N = \frac{N(N+1)}{2} \in \mathcal{O}\left(N^2\right)$

Sample recurrences and their solutions (contd.)

Example (Recurrence for doing one at a time with full sifting)

$$
\begin{aligned}
T(N) &= 1 && \text{for } N = 1 \\
T(N) &= T(N-1) + N && \text{for } N \geq 2
\end{aligned}
$$

- Linear but not homogenous of order 1
- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 2 = 1 + 2 = 3$
- $T(3) = T(2) + 3 = 1 + 2 + 3 = 6$
- $T(N) = T(N-1) + 1 = 1 + 2 + 3 + \ldots + N = \frac{N(N+1)}{2} \in \mathcal{O}\left(N^2\right)$
- Show that this recurrence captures the running time complexity of bubble/insertion/selection sort

Sample recurrences and their solutions (contd.)

Example (Recurrence for a Herculean task)

- $$\begin{aligned} T(N) &= 1 && \text{for } N = 1 \\ T(N) &= 2T(N-1) + 1 && \text{for } N \geq 2 \end{aligned}$$

Sample recurrences and their solutions (contd.)

Example (Recurrence for a Herculean task)

- $$\begin{aligned} T(N) &= 1 && \text{for } N = 1 \\ T(N) &= 2T(N-1) + 1 && \text{for } N \geq 2 \end{aligned}$$
- Linear but not homogenous of order 1

Sample recurrences and their solutions (contd.)

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- $$\begin{aligned} T(N) &= 1 && \text{for } N = 1 \\ T(N) &= 2T(N-1) + 1 && \text{for } N \geq 2 \end{aligned}$$
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- $T(1) = 1$

Sample recurrences and their solutions (contd.)

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- $$\begin{aligned} T(N) &= 1 && \text{for } N = 1 \\ T(N) &= 2T(N-1) + 1 && \text{for } N \geq 2 \end{aligned}$$
- Linear but not homogenous of order 1
- $T(1) = 1$
- $T(2) = 2T(1) + 1 = 2 + 1 = 3$

Sample recurrences and their solutions (contd.)

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- $$\begin{aligned} T(N) &= 1 && \text{for } N = 1 \\ T(N) &= 2T(N-1) + 1 && \text{for } N \geq 2 \end{aligned}$$
- Linear but not homogenous of order 1
- $T(1) = 1$
- $T(2) = 2T(1) + 1 = 2 + 1 = 3$
- $T(3) = 2T(2) + 1 = 6 + 1 = 7$

Sample recurrences and their solutions (contd.)

Example (Recurrence for a Herculean task)

- $$\begin{aligned} T(N) &= 1 && \text{for } N = 1 \\ T(N) &= 2T(N-1) + 1 && \text{for } N \geq 2 \end{aligned}$$
- Linear but not homogenous of order 1
- $T(1) = 1$
- $T(2) = 2T(1) + 1 = 2 + 1 = 3$
- $T(3) = 2T(2) + 1 = 6 + 1 = 7$
- $T(4) = 2T(3) + 1 = 14 + 1 = 15$

Sample recurrences and their solutions (contd.)

Example (Recurrence for a Herculean task)

- $$\begin{aligned} T(N) &= 1 && \text{for } N = 1 \\ T(N) &= 2T(N-1) + 1 && \text{for } N \geq 2 \end{aligned}$$
- Linear but not homogenous of order 1
- $T(1) = 1$
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- $T(3) = 2T(2) + 1 = 6 + 1 = 7$
- $T(4) = 2T(3) + 1 = 14 + 1 = 15$
- $T(N) = 2^N - 1 \Rightarrow T(N) \in \mathcal{O}\left(2^N\right)$

Sample recurrences and their solutions (contd.)

Example (Recurrence for a Herculean task)

- $$T(N) = 1 \qquad \text{for } N = 1$$
  $$T(N) = 2T(N-1) + 1 \qquad \text{for } N \geq 2$$
- Linear but not homogenous of order 1
- $T(1) = 1$
- $T(2) = 2T(1) + 1 = 2 + 1 = 3$
- $T(3) = 2T(2) + 1 = 6 + 1 = 7$
- $T(4) = 2T(3) + 1 = 14 + 1 = 15$
- $T(N) = 2^N - 1 \Rightarrow T(N) \in \mathcal{O}\left(2^N\right)$
- Show that this recurrence captures the running time complexity of the towers of Hanoi problem

Sample recurrences and their solutions (contd.)

Example (Recurrence for bank interest)

$$A_0 = P$$

$$A_n = A_{n-1} \underbrace{\left(1 + \frac{r}{100}\right)}_{\text{constant}}$$

Sample recurrences and their solutions (contd.)

Example (Recurrence for bank interest)

$$A_0 = P$$

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Sample recurrences and their solutions (contd.)

Example (Recurrence for bank interest)

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$$A_n = A_{n-1} \underbrace{\left(1 + \frac{r}{100}\right)}_{\text{constant}}$$

- linear homogeneous of order one
- $A_n = \left(1 + \frac{r}{100}\right)^n$

Sample recurrences and their solutions (contd.)

Example (Recurrence for bank interest)

$$A_0 = P$$
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- linear homogeneous of order one
- $A_n = \left(1 + \frac{r}{100}\right)^n$
- Exponential growth – not sustainable

Sample recurrences and their solutions (contd.)

Example (Recurrence for bank interest)

$$A_0 = P$$
$$A_n = A_{n-1} \underbrace{\left(1 + \frac{r}{100}\right)}_{\text{constant}}$$

- linear homogeneous of order one
- $A_n = \left(1 + \frac{r}{100}\right)^n$
- Exponential growth – not sustainable
- All banks curtail fixed deposits to a certain period of time

Sample recurrences and their solutions (contd.)

Example (Recurrence for eliminating half at a time)

$$T(N) = 1 \quad \text{for } N = 1$$
$$T(N) = T(N/2) + 1 \quad \text{for } N \geq 2$$

Sample recurrences and their solutions (contd.)

Example (Recurrence for eliminating half at a time)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = T(N/2) + 1 \qquad \text{for } N \geq 2$$

- Apply the method of iteration

Sample recurrences and their solutions (contd.)

Example (Recurrence for eliminating half at a time)

$$T(N) = 1 \quad \text{for } N = 1$$
$$T(N) = T(N/2) + 1 \quad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$

Sample recurrences and their solutions (contd.)

Example (Recurrence for eliminating half at a time)

$$T(N) = 1 \quad \text{for } N = 1$$
$$T(N) = T(N/2) + 1 \quad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 1 = 2$

Sample recurrences and their solutions (contd.)

Example (Recurrence for eliminating half at a time)

$$T(N) = 1 \quad \text{for } N = 1$$
$$T(N) = T(N/2) + 1 \quad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 1 = 2$
- $T(4) = T(2) + 1 = 3$

Sample recurrences and their solutions (contd.)

Example (Recurrence for eliminating half at a time)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = T(N/2) + 1 \qquad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 1 = 2$
- $T(4) = T(2) + 1 = 3$
- $T(2^N) = N + 1 \Rightarrow T(N) = \lg N + 1 \in \mathcal{O}(\lg N)$

Sample recurrences and their solutions (contd.)

Example (Recurrence for eliminating half at a time)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = T(N/2) + 1 \qquad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 1 = 2$
- $T(4) = T(2) + 1 = 3$
- $T(2^N) = N + 1 \Rightarrow T(N) = \lg N + 1 \in \mathcal{O}(\lg N)$
- Show that this recurrence captures the running time complexity of binary search

Sample recurrences and their solutions (contd.)

Example (Recurrence for eliminating half with full sifting)

$$T(N) = 1 \quad \text{for } N = 1$$
$$T(N) = T(N/2) + N \quad \text{for } N \geq 2$$

Sample recurrences and their solutions (contd.)

Example (Recurrence for eliminating half with full sifting)

$$T(N) = 1 \quad \text{for } N = 1$$
$$T(N) = T(N/2) + N \quad \text{for } N \geq 2$$

- Apply the method of iteration

Sample recurrences and their solutions (contd.)

Example (Recurrence for eliminating half with full sifting)

$$T(N) = 1 \quad \text{for } N = 1$$
$$T(N) = T(N/2) + N \quad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$

Sample recurrences and their solutions (contd.)

Example (Recurrence for eliminating half with full sifting)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = T(N/2) + N \qquad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 2 = 1 + 2 = 3$

Sample recurrences and their solutions (contd.)

Example (Recurrence for eliminating half with full sifting)

$$T(N) = 1 \quad \text{for } N = 1$$
$$T(N) = T(N/2) + N \quad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 2 = 1 + 2 = 3$
- $T(4) = T(2) + 4 = 3 + 4 = 7$

Sample recurrences and their solutions (contd.)

Example (Recurrence for eliminating half with full sifting)

$$T(N) = 1 \quad \text{for } N = 1$$
$$T(N) = T(N/2) + N \quad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 2 = 1 + 2 = 3$
- $T(4) = T(2) + 4 = 3 + 4 = 7$
- $T(8) = T(4) + 8 = 7 + 8 = 15$

Sample recurrences and their solutions (contd.)

Example (Recurrence for eliminating half with full sifting)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = T(N/2) + N \qquad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 2 = 1 + 2 = 3$
- $T(4) = T(2) + 4 = 3 + 4 = 7$
- $T(8) = T(4) + 8 = 7 + 8 = 15$
- $T(2^N) = 2^{N+1} - 1 \Rightarrow T(N) = 2N - 1 \in \mathcal{O}(N)$

Sample recurrences and their solutions (contd.)

Example (Recurrence for eliminating half with full sifting)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = T(N/2) + N \qquad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$
- $T(2) = T(1) + 2 = 1 + 2 = 3$
- $T(4) = T(2) + 4 = 3 + 4 = 7$
- $T(8) = T(4) + 8 = 7 + 8 = 15$
- $T(2^N) = 2^{N+1} - 1 \Rightarrow T(N) = 2N - 1 \in \mathcal{O}(N)$
- What procedure satisfies this recurrence?

Sample recurrences and their solutions (contd.)

Example (Recurrence for divide and conquer with full sifting)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = 2T(N/2) + N \qquad \text{for } N \geq 2$$

Sample recurrences and their solutions (contd.)

Example (Recurrence for divide and conquer with full sifting)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = 2T(N/2) + N \qquad \text{for } N \geq 2$$

- Apply the method of iteration

Sample recurrences and their solutions (contd.)

Example (Recurrence for divide and conquer with full sifting)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = 2T(N/2) + N \qquad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$

Sample recurrences and their solutions (contd.)

Example (Recurrence for divide and conquer with full sifting)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = 2T(N/2) + N \qquad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$
- $T(2) = 2T(1) + 2 = 2 + 2 = 4$

Sample recurrences and their solutions (contd.)

Example (Recurrence for divide and conquer with full sifting)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = 2T(N/2) + N \qquad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$
- $T(2) = 2T(1) + 2 = 2 + 2 = 4$
- $T(4) = 2T(2) + 4 = 8 + 4 = 12$

Sample recurrences and their solutions (contd.)

Example (Recurrence for divide and conquer with full sifting)

$$T(N) = 1 \quad \text{for } N = 1$$
$$T(N) = 2T(N/2) + N \quad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$
- $T(2) = 2T(1) + 2 = 2 + 2 = 4$
- $T(4) = 2T(2) + 4 = 8 + 4 = 12$
- $T(8) = 2T(4) + 8 = 24 + 8 = 32$

Sample recurrences and their solutions (contd.)

Example (Recurrence for divide and conquer with full sifting)

$$T(N) = 1 \quad \text{for } N = 1$$
$$T(N) = 2T(N/2) + N \quad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$
- $T(2) = 2T(1) + 2 = 2 + 2 = 4$
- $T(4) = 2T(2) + 4 = 8 + 4 = 12$
- $T(8) = 2T(4) + 8 = 24 + 8 = 32$
- $T(2^N) = N2^N \Rightarrow T(N) = N(1 + \lg N) \in \mathcal{O}(N \lg N)$

Sample recurrences and their solutions (contd.)

Example (Recurrence for divide and conquer with full sifting)

$$T(N) = 1 \qquad \text{for } N = 1$$
$$T(N) = 2T(N/2) + N \qquad \text{for } N \geq 2$$

- Apply the method of iteration
- $T(1) = 1$
- $T(2) = 2T(1) + 2 = 2 + 2 = 4$
- $T(4) = 2T(2) + 4 = 8 + 4 = 12$
- $T(8) = 2T(4) + 8 = 24 + 8 = 32$
- $T(2^N) = N2^N \Rightarrow T(N) = N(1 + \lg N) \in \mathcal{O}(N \lg N)$
- Show that this recurrence captures the running time complexity of mergesort and quicksort

Section outline

2. **Linear recurrences with constant coefficients**
   - Classification of some recurrences
   - Solving linear homogenous recurrences (LHR)
   - Combining satisfying sequences of LHRs
   - Master theorem (MT) for LHRs
   - Some applications of MT for LHRs
   - LHR master theorem proof
   - More applications of MT for LHRs

Classification of some recurrences

Some recurrences for classification

- Number of palindromes on the English alphabet: $P_n = 26P_{n-2}, n \geq 2$, $P_0 = 1$, $P_1 = 26$

Classification of some recurrences

Some recurrences for classification

- Number of palindromes on the English alphabet: $P_n = 26P_{n-2}, n \geq 2$, $P_0 = 1$, $P_1 = 26$ – linear homogeneous recurrence of order two
- Fibonacci sequence: $F_n = F_{n-1} + F_{n-2}$, $F_0 = F_1 = 1$

Classification of some recurrences

Some recurrences for classification

- Number of palindromes on the English alphabet: $P_n = 26P_{n-2}, n \geq 2$, $P_0 = 1$, $P_1 = 26$ – linear homogeneous recurrence of order two
- Fibonacci sequence: $F_n = F_{n-1} + F_{n-2}$, $F_0 = F_1 = 1$ – linear homogeneous recurrence of order two
- Factorials: $f_n = nf_{n-1}$

Classification of some recurrences

Some recurrences for classification

- Number of palindromes on the English alphabet: $P_n = 26P_{n-2}, n \geq 2$, $P_0 = 1$, $P_1 = 26$ – linear homogeneous recurrence of order two
- Fibonacci sequence: $F_n = F_{n-1} + F_{n-2}$, $F_0 = F_1 = 1$ – linear homogeneous recurrence of order two
- Factorials: $f_n = nf_{n-1}$ – non-linear homogeneous recurrence of order one

Classification of some recurrences (contd.)

Some recurrences for classification (contd.)

- Derangements: $D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$; $D_1 = 0$, $D_2 = 1$ or $D_0 = 1$, $D_1 = 0$ – homogeneous recurrence of order two with non-constant coefficients
- Catalan numbers: $C_n = C_0C_{n-1} + C_1C_{n-2} + \ldots + C_{n-2}C_1 + C_{n-1}C_0$

Classification of some recurrences (contd.)

Some recurrences for classification (contd.)

- Derangements: $D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$; $D_1 = 0$, $D_2 = 1$ or $D_0 = 1$, $D_1 = 0$ – homogeneous recurrence of order two with non-constant coefficients
- Catalan numbers: $C_n = C_0 C_{n-1} + C_1 C_{n-2} + \ldots + C_{n-2} C_1 + C_{n-1} C_0$ – non-linear recurrence
- DC: $T_n = T_{\lfloor \frac{n}{2} \rfloor} + T_{\lceil \frac{n}{2} \rceil} + cn + d$

Classification of some recurrences (contd.)

Some recurrences for classification (contd.)

- Derangements: $D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$; $D_1 = 0$, $D_2 = 1$ or $D_0 = 1$, $D_1 = 0$ – homogeneous recurrence of order two with non-constant coefficients
- Catalan numbers: $C_n = C_0 C_{n-1} + C_1 C_{n-2} + \ldots + C_{n-2} C_1 + C_{n-1} C_0$ – non-linear recurrence
- DC: $T_n = T_{\lfloor \frac{n}{2} \rfloor} + T_{\lceil \frac{n}{2} \rceil} + cn + d$ – linear recurrence, but not of constant order

Solving linear homogenous recurrences (LHR)

- Form: $T_n = a_1 T_{n-1} + a_2 T_{n-2} + \ldots + a_{k-1} T_{n-k+1} + a_k T_{n-k}$, $a_k \neq 0$
- $$\begin{bmatrix} T_n \\ T_{n-1} \\ \vdots \\ T_{n-k+2} \\ T_{n-k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & a_2 & \ldots & 0 & a_{k-1} & a_k \\ 1 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} T_{n-1} \\ T_{n-2} \\ \vdots \\ T_{n-k+1} \\ T_{n-k} \end{bmatrix}$$

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- $$\begin{bmatrix} T_n \\ T_{n-1} \\ \vdots \\ T_{n-k+2} \\ T_{n-k+1} \end{bmatrix} = A \begin{bmatrix} T_{n-1} \\ T_{n-2} \\ \vdots \\ T_{n-k+1} \\ T_{n-k} \end{bmatrix}$$

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- Note that $T_0, T_1, \ldots, T_{k-1}$ are (constant) initial values
- Also, matrix A is invertible: $|A| = (-1)^{k-1} a_k$ (as $a_k \neq 0$)
- So, A has rank k and has one or more basis of k linearly independent vectors

Solving linear homogenous recurrences (contd.)

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- $D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_k \end{bmatrix} \Rightarrow D^n = \begin{bmatrix} \lambda_1^n & 0 & \ldots & 0 \\ 0 & \lambda_2^n & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_k^n \end{bmatrix}$ (nice property)

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- How to find D?

Solving linear homogenous recurrences (contd.)

- Let $e_1, e_2, \ldots, e_k$ be linearly independent eigen vectors of A (of rank k)

Solving linear homogenous recurrences (contd.)

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- Thus A has been expressed as $A = PDP^{-1}$, but need to determine D

Solving linear homogenous recurrences (contd.)

- Compute the eigenvalues of the matrix A by solving $|A - \lambda I| = 0$

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- $(a_1 - \lambda)(-\lambda)^{k-1} - a_2(-\lambda)^{k-2} + a_3(-\lambda)^{k-3} - \ldots + (-1)^{k-1} a_k = 0$

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- $\underbrace{\lambda^k - a_1\lambda^{k-1} - a_2\lambda^{k-2} - \dots - a_k}_{\text{characteristic polynomial } \chi_A(\lambda)} = 0$
- Roots of $\chi_A(\lambda)$ are the eigen values of A

Solving linear homogenous recurrences (contd.)

- $$\begin{bmatrix} T_{n+k} \\ T_{n+k-1} \\ \vdots \\ T_{n+1} \\ T_n \end{bmatrix} = \underbrace{\underbrace{\begin{bmatrix} \lambda_1^n e_{11} & \lambda_2^n e_{21} & \dots & \lambda_k^n e_{k1} \\ \lambda_1^n e_{12} & \lambda_2^n e_{22} & \dots & \lambda_k^n e_{k2} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^n e_{1k} & \lambda_2^n e_{2k} & \dots & \lambda_k^n e_{kk} \end{bmatrix}}_{PD^n} P^{-1}}_{A^n=(PD^n)P^{-1}} \begin{bmatrix} T_{k-1} \\ T_{k-2} \\ \vdots \\ T_1 \\ T_0 \end{bmatrix} \text{ where}$$

$$P^{-1} = \begin{bmatrix} e_{11} & e_{21} & \dots & e_{k1} \\ e_{12} & e_{22} & \dots & e_{k2} \\ \vdots & \vdots & \vdots & \vdots \\ e_{1k} & e_{2k} & \dots & e_{kk} \end{bmatrix}^{-1} = \begin{bmatrix} d_{11} & d_{21} & \dots & d_{k1} \\ d_{12} & d_{22} & \dots & d_{k2} \\ \vdots & \vdots & \vdots & \vdots \\ d_{1k} & d_{2k} & \dots & d_{kk} \end{bmatrix} \text{ (say)}$$

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$$P^{-1} = \begin{bmatrix} e_{11} & e_{21} & \ldots & e_{k1} \\ e_{12} & e_{22} & \ldots & e_{k2} \\ \vdots & \vdots & \vdots & \vdots \\ e_{1k} & e_{2k} & \ldots & e_{kk} \end{bmatrix}^{-1} = \begin{bmatrix} d_{11} & d_{21} & \ldots & d_{k1} \\ d_{12} & d_{22} & \ldots & d_{k2} \\ \vdots & \vdots & \vdots & \vdots \\ d_{1k} & d_{2k} & \ldots & d_{kk} \end{bmatrix} \text{ (say)}$$
- Thus, T_n will be a linear combination of $\lambda_\jmath^n$, $1 \leq \jmath \leq k$

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- Thus, T_n will be a linear combination of λ_j^n, $1 \leq j \leq k$
- $$T_n = \begin{matrix} [\lambda_1^n (e_{1k} d_{11}) + \lambda_2^n (e_{2k} d_{12}) + \ldots + \lambda_k^n (e_{kk} d_{1k})] \, T_{k-1} & + \\ [\lambda_1^n (e_{1k} d_{21}) + \lambda_2^n (e_{2k} d_{22}) + \ldots + \lambda_k^n (e_{kk} d_{2k})] \, T_{k-2} & + \quad \dots \quad + \\ [\lambda_1^n (e_{1k} d_{k1}) + \lambda_2^n (e_{2k} d_{k2}) + \ldots + \lambda_k^n (e_{kk} d_{kk})] \, T_0 & \end{matrix}$$

Combining satisfying sequences of LHRs

- It's important to compute the roots of the characteristic equation!
- Note that λ^n satisfies the recurrence
- We shall use this observation to derive the solution without computing the eigen vectors

- If sequences $p(n)$ and $q(n)$ are both satisfy the LHR $T_n = a_1 T_{n-1} + a_2 T_{n-2} + \ldots + a_k T_{n-k}$, then $r(n) = bp(n) + cq(n)$ is also satisfies the LHR for all $b, c \in \mathbb{R}$

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- $r(n) = bp(n) + cq(n)$ – prove by substitution
$$= b(a_1 p(n-1) + a_2 p(n-2) + \ldots + a_k p(n-k)) +$$
$$c(a_1 q(n-1) + a_2 q(n-2) + \ldots + a_k q(n-k))$$

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- $$\begin{aligned} r(n) &= bp(n) + cq(n) \text{ – prove by substitution} \\ &= b(a_1 p(n-1) + a_2 p(n-2) + \ldots + a_k p(n-k)) + \\ &\quad c(a_1 q(n-1) + a_2 q(n-2) + \ldots + a_k q(n-k)) \\ &= a_1(bp(n-1) + cq(n-1)) + a_2(bp(n-2) + cq(n-2)) \\ &\quad + \ldots + a_0(bp(n-k) + cq(n-k)) \end{aligned}$$

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$$
\begin{aligned}
r(n) &= bp(n) + cq(n) \\
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&\quad c(a_1 q(n-1) + a_2 q(n-2) + \ldots + a_k q(n-k)) \\
&= a_1(bp(n-1) + cq(n-1)) + a_2(bp(n-2) + cq(n-2)) \\
&\quad + \ldots + a_0(bp(n-k) + cq(n-k)) \\
&= a_1 r(n-1) + a_2 r(n-2) + \ldots + a_k r(n-k)
\end{aligned}
$$

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$$\begin{aligned} &= b(a_1 p(n-1) + a_2 p(n-2) + \ldots + a_k p(n-k)) + \\ &\quad c(a_1 q(n-1) + a_2 q(n-2) + \ldots + a_k q(n-k)) \\ &= a_1(bp(n-1) + cq(n-1)) + a_2(bp(n-2) + cq(n-2)) \\ &\quad + \ldots + a_0(bp(n-k) + cq(n-k)) \\ &= a_1 r(n-1) + a_2 r(n-2) + \ldots + a_k r(n-k) \end{aligned}$$

- From now ρ will denote the roots instead of λ

Testing and combining sequences

Example (A recurrence of order 2)

Consider $F_n = F_{n-1} + F_{n-2}$

- Does $T_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$ satisfy this recurrence?

Testing and combining sequences

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$$\text{LHS} = \left(\frac{1+\sqrt{5}}{2}\right)^n, \text{RHS} = \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}$$

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$$= \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} \left[\frac{1+\sqrt{5}}{2} + 1\right] =$$

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$$=\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left[\frac{1+\sqrt{5}}{2}+1\right]=$$

$$=\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\frac{1+2\sqrt{5}+5}{4}=\left(\frac{1+\sqrt{5}}{2}\right)^n$$

Testing and combining sequences (contd.)

Example (A recurrence of order 2 (contd.))

Consider $F_n = F_{n-1} + F_{n-2}$

- Does $F_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$ satisfy this recurrence?

Testing and combining sequences (contd.)

Example (A recurrence of order 2 (contd.))

Consider $F_n = F_{n-1} + F_{n-2}$

- Does $F_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$ satisfy this recurrence?

$$\text{LHS}= \left(\frac{1-\sqrt{5}}{2}\right)^n, \text{RHS}= \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}$$

Testing and combining sequences (contd.)

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$$= \left(\frac{1-\sqrt{5}}{2}\right)^{n-2} \left[\frac{1-\sqrt{5}}{2} + 1\right] =$$

Testing and combining sequences (contd.)

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$$= \left(\frac{1-\sqrt{5}}{2}\right)^{n-2} \left[\frac{1-\sqrt{5}}{2} + 1\right] =$$

$$= \left(\frac{1-\sqrt{5}}{2}\right)^{n-2} \frac{1-2\sqrt{5}+5}{4} = \left(\frac{1-\sqrt{5}}{2}\right)^n$$

- $F_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$ will be a solution to the LHR

Testing and combining sequences (contd.)

Example (A recurrence of order 2 (contd.))

- Determine c_1 and c_2 so that $F_0 = 0$, $F_1 = 1$
- $c_1 + c_2 = 0$, so $c_1 = -c_2$ [for $n = 0$]
- $c_1 \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \left(\frac{1-\sqrt{5}}{2}\right) = c_1\sqrt{5} = 1$ [for $n = 1$]
- $c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\sqrt{5}}$

Testing and combining sequences (contd.)

Example (A recurrence of order 2 (contd.))

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- $c_1 \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \left(\frac{1-\sqrt{5}}{2}\right) = c_1\sqrt{5} = 1$ [for $n = 1$]
- $c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\sqrt{5}}$
- What if $F_0 = 0$ and $F_2 = 1$ had been provided instead?

Testing and combining sequences (contd.)

Example (Another recurrence of order 2)

Consider $T_n = 2aT_{n-1} - a^2 T_{n-2}$

- Does $T_n = a^n$ satisfy this recurrence?

Testing and combining sequences (contd.)

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Consider $T_n = 2aT_{n-1} - a^2 T_{n-2}$

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- LHS$= a^n$, RHS$= 2aa^{n-1} - a^2 a^{n-2} = a^n$

Testing and combining sequences (contd.)

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- Does $T_n = na^n$ satisfy this recurrence?

Testing and combining sequences (contd.)

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- LHS$= a^n$, RHS$= 2aa^{n-1} - a^2 a^{n-2} = a^n$
- Does $T_n = na^n$ satisfy this recurrence?
- LHS$= na^n$, RHS$= 2ana^{n-1} - a^2 na^{n-2} = na^n$
- $T_n = c_1 a^n + c_2 na^n$ will be the solution
- c_1 and c_2 can be determined from the initial conditions

Master theorem (MT) for LHRs

Theorem (Master theorem (MT) for LHRs)

- *Consider the LHR with constant coefficients* $a_0, a_1, \ldots, a_{k-1}$

$$T_n = a_1 T_{n-1} + a_2 T_{n-2} + \ldots + a_k T_{n-k} \tag{4.1}$$

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- *Its characteristic polynomial has* k *roots (some may have multiplicity):*

$$\chi_A(x) = x^k - a_1 x^{k-1} - a_2 x^{k-2} - \ldots - a_k \quad (4.2)$$

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- *Let the roots be* ρ_1 *with multiplicity* m_1, ρ_2 *with multiplicity* m_2, $\ldots$, ρ_t *with multiplicity* m_t, *so that* $m_1 + m_2 + \ldots + m_t = k$
- *Recurrence 4.1 is satisfied by* $T_n = \sum_{\imath=1}^{t} \sum_{\jmath=0}^{m_\imath - 1} c_{\imath\jmath} n^\jmath \rho_\imath^n$ *(*$c_{\imath\jmath}$ *being constants)*
- *Elaborated as:* $T_n = \left(c_{1,0} + c_{1,1} n + \ldots + c_{1,m_1-1} n^{m_1-1}\right)\rho_1^n + \left(c_{2,0} + c_{2,1} n + \ldots + c_{2,m_1-1} n^{m_2-1}\right)\rho_2^n + \ldots + \left(c_{t,0} + c_{t,1} n + \ldots + c_{t,m_1-1} n^{m_t-1}\right)\rho_t^n,$

Some applications of MT for LHRs

Example

- $T_n = aT_{n-1}$

Some applications of MT for LHRs

Example

- $T_n = aT_{n-1}$, $\chi_A(x) = x - a = 0$, $c = T_0$, $T_n = T_0 a^n$

Some applications of MT for LHRs

Example

- $T_n = aT_{n-1}$, $\chi_A(x) = x - a = 0$, $c = T_0$, $T_n = T_0 a^n$
- $P_n = 26P_{n-2}$, $\chi_A(x) = x^2 - 26 = 0$, $\{\rho\} : \pm\sqrt{26}$,
  $P_n = c_1(\sqrt{26})^n + c_2(-\sqrt{26})^n$

Some applications of MT for LHRs

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- $T_n = aT_{n-1}$, $\chi_A(x) = x - a = 0$, $c = T_0$, $T_n = T_0 a^n$
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  Using $P_0 = 1$ and $P_1 = 26$, $c_1 = \dfrac{1+\sqrt{26}}{2}$, $c_2 = \dfrac{1-\sqrt{26}}{2}$

Some applications of MT for LHRs

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- $T_n = aT_{n-1}$, $\chi_A(x) = x - a = 0$, $c = T_0$, $T_n = T_0 a^n$
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  $P_n = c_1(\sqrt{26})^n + c_2(-\sqrt{26})^n$
  Using $P_0 = 1$ and $P_1 = 26$, $c_1 = \frac{1+\sqrt{26}}{2}$, $c_2 = \frac{1-\sqrt{26}}{2}$
  $$P_n = \left(\frac{1+\sqrt{26}}{2}\right)(\sqrt{26})^n + \left(\frac{1-\sqrt{26}}{2}\right)(-\sqrt{26})^n$$

Some applications of MT for LHRs (contd.)

Example

- $F_n = F_{n-1} + F_{n-2}$, $\chi(x) = x^2 - x - 1 = 0$

Some applications of MT for LHRs (contd.)

Example

- $F_n = F_{n-1} + F_{n-2}$, $\chi(x) = x^2 - x - 1 = 0$, $\rho_1 = \frac{1+\sqrt{5}}{2}$, $\rho_2 = \frac{1-\sqrt{5}}{2}$, $F_n = c_1\rho_1^n + c_2\rho_2^n$

Some applications of MT for LHRs (contd.)

Example

- $F_n = F_{n-1} + F_{n-2}$, $\chi(x) = x^2 - x - 1 = 0$, $\rho_1 = \dfrac{1+\sqrt{5}}{2}$, $\rho_2 = \dfrac{1-\sqrt{5}}{2}$, $F_n = c_1\rho_1^n + c_2\rho_2^n$

  Using $F_0 = 0$ and $F_1 = 1$, $c_1 = \dfrac{1}{\sqrt{5}}$, $c_2 = -\dfrac{1}{\sqrt{5}}$

Some applications of MT for LHRs (contd.)

Example

- $F_n = F_{n-1} + F_{n-2}$, $\chi(x) = x^2 - x - 1 = 0$, $\rho_1 = \dfrac{1+\sqrt{5}}{2}$, $\rho_2 = \dfrac{1-\sqrt{5}}{2}$, $F_n = c_1\rho_1^n + c_2\rho_2^n$

  Using $F_0 = 0$ and $F_1 = 1$, $c_1 = \dfrac{1}{\sqrt{5}}$, $c_2 = -\dfrac{1}{\sqrt{5}}$

  $$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

LHR master theorem proof

- As $\chi_A(x) = x^k - a_1x^{k-1} - a_2x^{k-2} - \ldots - a_k = 0$ has as a root ρ with multiplicity m, we may express $\chi_A(x)$ as

$$\chi_A(x) = (x - \rho)^m \mu(x), \ \mu(x) \neq 0 \tag{6.1}$$

- $\chi_A(x)$ (in eq 6.1) may be differentiated repeatedly
- $\chi'_A(x) = m(x - \rho)^{m-1}\mu(x) + (x - \rho)^m \mu'(x)$

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- Likewise, $\chi''_A(\rho) = \ldots = \chi_A^{(m-1)}(\rho) = 0$
- Now, for $n \geq k$ we have by multiplying $\chi_A(x)$ [eq 4.2] by x^{n-k}:

$$x^{n-k}\chi_A(x) = x^n - a_1x^{n-1} - a_2x^{n-2} - \ldots - a_kx^{n-k} \tag{6.2}$$

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$$x^{n-k}\chi_A(x) = x^n - a_1 x^{n-1} - a_2 x^{n-2} - \ldots - a_k x^{n-k} \tag{6.2}$$

- On differentiating eq 6.2 wrt x and then multiplying with x, we get: $(n-k)x^{n-k}\chi_A(x) + x^{n-k+1}\chi'_A(x) =$ $nx^n - a_1(n-1)x^{n-1} - a_2(n-2)x^{n-1} - \ldots - a_k(n-k)x^{n-k}$

LHR master theorem proof (contd.)

- Differentiating again wrt x and then multiplying with x, we get:
  $(n-k)^2 x^{n-k}\chi_A(x)+[(n-k)+(n-k+1)]x^{n-k}\chi_A'(x)+x^{n-k+1}\chi_A''(x) =$
  $(n-k)^2 x^{n-k}\chi_A(x) + (2n-2k+1)x^{n-k}\chi_A'(x) + x^{n-k+1}\chi_A''(x) =$
  $n^2x^n - a_1(n-1)^2x^{n-1} - a_2(n-2)^2x^{n-2} - \ldots - a_k(n-k)^2x^{n-k}$
- Repeating this process of differentiating wrt x and then multiplying by x, s times we get:
  $f_0(x)\chi_A(x) + f_1(x)\chi_A'(x) + \ldots + f_s(x)\chi_A^{(s)}(x) =$
  $n^s x^n - a_1(n-1)^s x^{n-1} - a_2(n-2)^s x^{n-2} - \ldots - a_k(n-k)^s x^{n-k}$,
  where $f_0, \ldots, f_s$, are all polynomials in x
- Now, substituting $x = \rho$ and knowing that
  $\chi_A(\rho) = \chi_A'(\rho) = \ldots = \chi_A^{(s)}(\rho) = 0$, we get
  $n^s\rho^n - a_1(n-1)^s\rho^{n-1} - a_2(n-2)^s\rho^{n-2} - \ldots - a_k(n-k)^s\rho^{n-k} = 0$

LHR master theorem proof (contd.)

- This establishes that $T_n = n^s \rho^n$, $s = 0, \ldots, m-1$, m being the multiplicity of ρ, satisfies eq 4.1 [LHR]

LHR master theorem proof (contd.)

- This establishes that $T_n = n^s \rho^n$, $s = 0, \ldots, m-1$, m being the multiplicity of ρ, satisfies eq 4.1 [LHR]
- Let there t distinct roots with multiplicities $m_1, m_2, \ldots, m_t$
- So does any linear combination of $n^{s_1}\rho_1^n$, $s_1 = 0, \ldots, m_1 - 1$, $n^{s_2}\rho_2^n$, $s_2 = 0, \ldots, m_2 - 1$, $\ldots$, $n^{s_t}\rho_t^n$, $s_t = 0, \ldots, m_t - 1$
- Values of the coeefficients may be derived from the initial conditions

LHR master theorem proof (contd.)

- The initial conditions $T_0, T_1, \ldots, T_{k-1}$ may be equated with the solutions obtained as: $w_n = \sum_{\imath=1}^{t} \sum_{\jmath=0}^{m_\imath - 1} c_{\imath\jmath} n^\jmath \rho_\imath^n$,

LHR master theorem proof (contd.)

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- Leads to a system of linear equations for the t distinct roots with multiplicities $m_1, m_2, \ldots, m_t$:

$$
\begin{aligned}
\left. \sum_{\imath=1}^{t} \sum_{\jmath=0}^{m_\imath - 1} c_{\imath\jmath} n^\jmath \rho_\imath^n \right|_{n=0} &= T_0 \\
\left. \sum_{\imath=1}^{t} \sum_{\jmath=0}^{m_\imath - 1} c_{\imath\jmath} n^\jmath \rho_\imath^n \right|_{n=1} &= T_1 \\
\vdots \qquad\qquad & \vdots \quad \vdots \\
\left. \sum_{\imath=1}^{t} \sum_{\jmath=0}^{m_\imath - 1} c_{\imath\jmath} n^\jmath \rho_\imath^n \right|_{n=k-1} &= T_{k-1}
\end{aligned}
$$

LHR master theorem proof (contd.)

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\vdots \qquad\qquad &\vdots \quad \vdots \\
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\end{aligned}
$$

- For the coefficient matrix for this system of linear equations
- Number of rows: k
- Number of columns on the LHS: $m_1 + m_2 + \ldots + m_t = k$
- The coefficient matrix is invertible, so the constants can be uniquely determined

More applications of MT for LHRs

Example

Consider $T_n = 2T_{n-1} + T_{n-2} - 2T_{n-3}$ with $T_0 = 0, T_1 = 1, T_2 = 2$

- Characteristic equation: $\chi(x) = x^3 - 2x^2 - x + 2 = 0$,
- $\chi(x) = (x-1)(x+1)(x-2) = 0$; form of general solution is:

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- $T_n = c_2 2^n + c_1(-1)^n + c_0$; applying initial conditions we get:

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- $T_n = c_2 2^n + c_1(-1)^n + c_0$; applying initial conditions we get:
- $c_2 + c_1 + c_0 = 0$, $2c_2 - c_1 + c_0 = 1$, $4c_2 + c_1 + c_0 = 2$
- $c_2 = \frac{2}{3}, c_1 = -\frac{1}{6}, c_0 = -\frac{1}{2}$, so T_n is:

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- $\chi(x) = (x-1)(x+1)(x-2) = 0$; form of general solution is:
- $T_n = c_2 2^n + c_1(-1)^n + c_0$; applying initial conditions we get:
- $c_2 + c_1 + c_0 = 0, 2c_2 - c_1 + c_0 = 1, 4c_2 + c_1 + c_0 = 2$
- $c_2 = \frac{2}{3}, c_1 = -\frac{1}{6}, c_0 = -\frac{1}{2}$, so T_n is:
- $T_n = \frac{2}{3}2^n - \frac{1}{6}(-1)^n - \frac{1}{2} = \frac{2^{n+1}}{3} + \frac{(-1)^{n+1}}{6} - \frac{1}{2}, \forall n \in \mathbb{N}_0$

Some application of MT for LHRs (contd.)

Example

Consider $T_n = T_{n-1} + 4T_{n-3}$ with $T_0 = 0, T_1 = 1, T_2 = 2$

- Characteristic equation: $\chi(x) = x^3 - x^2 - 4 = 0,$
- $\chi(x) = (x-3)(x^2+x+2) = 0, \{\rho\}:$
  $\rho_0 = 2, \rho_1 = \frac{-1+\imath\sqrt{7}}{2}, \rho_2 = \frac{-1-\imath\sqrt{7}}{2}$

Some application of MT for LHRs (contd.)

Example

Consider $T_n = T_{n-1} + 4T_{n-3}$ with $T_0 = 0, T_1 = 1, T_2 = 2$

- Characteristic equation: $\chi(x) = x^3 - x^2 - 4 = 0,$
- $\chi(x) = (x-3)(x^2+x+2) = 0, \{\rho\}$:
  $\rho_0 = 2, \rho_1 = \frac{-1+\imath\sqrt{7}}{2}, \rho_2 = \frac{-1-\imath\sqrt{7}}{2}$
- $T_n = c_2\left(\frac{-1-\imath\sqrt{7}}{2}\right)^n + c_1\left(\frac{-1+\imath\sqrt{7}}{2}\right)^n + c_0 2^n$

Some application of MT for LHRs (contd.)

Example

Consider $T_n = T_{n-1} + 4T_{n-3}$ with $T_0 = 0$, $T_1 = 1$, $T_2 = 2$

- Characteristic equation: $\chi(x) = x^3 - x^2 - 4 = 0$,
- $\chi(x) = (x-3)(x^2+x+2) = 0$, $\{\rho\}$: $\rho_0 = 2, \rho_1 = \frac{-1+\imath\sqrt{7}}{2}, \rho_2 = \frac{-1-\imath\sqrt{7}}{2}$
- $T_n = c_2 \left(\frac{-1-\imath\sqrt{7}}{2}\right)^n + c_1 \left(\frac{-1+\imath\sqrt{7}}{2}\right)^n + c_0 2^n$
- $c_2 + c_1 + c_0 = 0$, $c_2 \frac{-1-\imath\sqrt{7}}{2} + c_1 \frac{-1-\imath\sqrt{7}}{2} + 2c_0 = 1$, $c_2 \frac{-3+\imath\sqrt{7}}{2} + c_1 \frac{-3-\imath\sqrt{7}}{2} + 4c_0 = 2$, applying initial conditions
- $c_2 = -\frac{1}{16\sqrt{7}}\left(3\sqrt{7}+\imath\right), c_1 = -\frac{1}{16\sqrt{7}}\left(3\sqrt{7}-\imath\right), c_0 = \frac{3}{8}$

Some application of MT for LHRs (contd.)

Example

Consider $T_n = T_{n-1} + 4T_{n-3}$ with $T_0 = 0$, $T_1 = 1$, $T_2 = 2$

- Characteristic equation: $\chi(x) = x^3 - x^2 - 4 = 0$,
- $\chi(x) = (x-3)(x^2+x+2) = 0$, $\{\rho\}$: $\rho_0 = 2, \rho_1 = \frac{-1+\imath\sqrt{7}}{2}, \rho_2 = \frac{-1-\imath\sqrt{7}}{2}$
- $T_n = c_2\left(\frac{-1-\imath\sqrt{7}}{2}\right)^n + c_1\left(\frac{-1+\imath\sqrt{7}}{2}\right)^n + c_0 2^n$
- $c_2 + c_1 + c_0 = 0$, $c_2\frac{-1-\imath\sqrt{7}}{2} + c_1\frac{-1-\imath\sqrt{7}}{2} + 2c_0 = 1$, $c_2\frac{-3+\imath\sqrt{7}}{2} + c_1\frac{-3-\imath\sqrt{7}}{2} + 4c_0 = 2$, applying initial conditions
- $c_2 = -\frac{1}{16\sqrt{7}}\left(3\sqrt{7}+\imath\right), c_1 = -\frac{1}{16\sqrt{7}}\left(3\sqrt{7}-\imath\right), c_0 = \frac{3}{8}$
- $T_n = c_0\rho_0^n + c_1\rho_1^n + c_2\rho_2^n, \forall n \in \mathbb{N}_0$

Some application of MT for LHRs (contd.)

Example (Strings with no consecutive vowels $\{a, e, i, o, u\}$) (NCV))

How many NCV strings (S_n) of n characters over the alphabet $\{a, b, \ldots, z\}$?

- Let V_n be the number of NCVs of length n ending in a vowel
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- $\{\rho\} = \left\{\frac{1}{2}\left(21 \pm \sqrt{861}\right)\right\}$, $C_1 = 21$, $C_0 = 1$, $C_2 = 546$

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- $S_n = C_n + V_n = C_n + 5C_{n-1}$, $S_2 = 651$, $S_1 = 26$, $S_0 = 1$

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- $S_n = C_n + V_n = C_n + 5C_{n-1}$, $S_2 = 651$, $S_1 = 26$, $S_0 = 1$
- $S_n = \frac{378+13\sqrt{861}}{\sqrt{861}}\left(\frac{21+\sqrt{861}}{2}\right)^{n-1} + \frac{-378+13\sqrt{861}}{\sqrt{861}}\left(\frac{21-\sqrt{861}}{2}\right)^{n-1}$

Some application of MT for LHRs (contd.)

Example (Number of NCVs for the English alphabet (contd.))

- $T_n = 2T_{n-1} - 4T_{n-2} + 8T_{n-3}$, $n \geq 3$, $T_0 = 1$, $T_1 = 1$, $T_2 = 1$

Some application of MT for LHRs (contd.)

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- $\chi(x) = x^3 - 2x^2 + 4x - 8 = (x-2)(x^2+4) = 0$,
  $\{\rho\} = \{2, 2\imath, -2\imath\}$ $T_n = c_1 2^n + c_2(2\imath)^n + c_3(-2\imath)^n$

Some application of MT for LHRs (contd.)

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  - $a_0 = 1 : c_1 2^0 + c_2(2\imath)^0 + c_3(-2\imath)^0 = c_1 + c_2 + c_3 = 1$
  - $a_1 = 1 : c_1 2^1 + c_2(2\imath)^1 + c_3(-2\imath)^1 = 2c_1 + 2\imath c_2 - 2\imath c_3 = 0$
  - $a_2 = 1 : c_1 2^2 + c_2(2\imath)^2 + c_3(-2\imath)^2 = 4c_1 - 4c_2 + 4c_3 = 1$

Some application of MT for LHRs (contd.)

Example (Number of NCVs for the English alphabet (contd.))

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  - $a_2 = 1 : c_1 2^2 + c_2(2\imath)^2 + c_3(-2\imath)^2 = 4c_1 - 4c_2 + 4c_3 = 1$
- $c_1 = \frac{5}{8}, c_2 = \frac{3}{16} + \imath\frac{1}{16}, c_3 = \frac{3}{16} - \imath\frac{1}{16}$
- $T_n = \frac{5}{8}2^n + \left(\frac{3}{16} + \imath\frac{1}{16}\right)(2\imath)^n + \left(\frac{3}{16} - \imath\frac{1}{16}\right)(-2\imath)^n$

Section outline

3 **Linear non-homogeneous recurrences**

- LNHR examples
- Solving LNHRs via LHRs and a particular solution

LNHR examples

Example (Strings with vowels in consecutive positions)

How many strings are there of length n over the alphabet $\{a, b, \ldots, z\}$, in which two vowels occur in some consecutive positions?

- The string can start with a consonant, in that case we would only be interested in a string of length $n-1$ containing two consecutive vowels to follow, contribution

LNHR examples

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- The string can start with two vowels, in that case the follwing string of length $n-2$ is unrestricted, contribution $5 \cdot 5 \cdot 26^{n-2}$
- $T_n = 21T_{n-1} + 105T_{n-2} + 25 \cdot 26^{n-2}$ – LNHR of order 2

LNHR example (contd.)

Example (Strings with vowels in consecutive positions (contd.))

Converting to LHR

- $T_n = 21T_{n-1} + 105T_{n-2} + 25 \cdot 26^{n-2}$
- $T_{n-1} = 21T_{n-2} + 105T_{n-3} + 25 \cdot 26^{n-3}$

$\therefore\ 26T_{n-1} = 21 \cdot 26T_{n-2} + 105 \cdot 26T_{n-3} + 25 \cdot 26^{n-2}$, subtracting,

- $T_n = 47T_{n-1} - 441T_{n-2} - 105T_{n-3}$, $T_0 = 0$, $T_1 = 0$, $T_2 = 25$
- $\chi(x) = x^3 - 47x^2 + 441x + 2730 = 0$ – this is an LHR
- Note that $\chi(26) = 0$, so $\chi(x) = (x - 26)p(x)$
- Dividing $\chi(x)$ by $x - 26$, $p(x) = x^2 - 21x - 105$, so $\chi(x) = (x - 26)\left(x^2 - 21x - 105\right)$

A trick solution utilising $T_n + S_n = 26^n$

So, $T_n = 26^n - \frac{31+\sqrt{861}}{2\sqrt{861}}\left(\frac{21+\sqrt{861}}{2}\right)^n - \frac{-31+\sqrt{861}}{2\sqrt{861}}\left(\frac{21-\sqrt{861}}{2}\right)^n$

Solving LNHRs via LHRs and a particular solution

- Let $T_n = U_n$ be a general solution of LNHR eq 1.1

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- Now $W_n = U_n - V_n =$
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- The initial conditions $\{T_\imath = t_\imath\}$ for T_n may be used to obtain initial conditions $\{W_\imath = t_\imath - g(\imath)\}$ for W_n

PS of LNHR, $g(n) = (b_r n^r + \ldots + b_1 n + b_0)\sigma^n$

- In $T_n = a_1 T_{n-1} + a_2 T_{n-2} + \ldots + a_k T_{n-k} + g(n)$ (eq 1.1), let

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- Let $\chi_A(x)$ be the characteristic equation; define

$$\mu = \begin{cases} 0 & \sigma \text{ is not a root of } \chi_A(x) \\ m & \sigma \text{ is a root of multiplicity } m \text{ of } \chi_A(x) \end{cases}$$

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- Let $\chi_A(x)$ be the characteristic equation; define

$$\mu = \begin{cases} 0 & \sigma \text{ is not a root of } \chi_A(x) \\ m & \sigma \text{ is a root of multiplicity } m \text{ of } \chi_A(x) \end{cases}$$

- Then $V_n = n^\mu (d_r n^r + d_{r-1} n^{r-1} + \ldots + d_1 n + d_0)\sigma^n$ may be determined as a particular solution (PS) of LNHR eqn 1.1, $d_\imath$ are complex (or real) constants which have to be determined

PS of LNHR, $g(n) = (b_r n^r + \ldots + b_1 n + b_0)\sigma^n$

- In $T_n = a_1 T_{n-1} + a_2 T_{n-2} + \ldots + a_k T_{n-k} + g(n)$ (eq 1.1), let

  $g(n) = (b_r n^r + b_{r-1} n^{r-1} + \ldots + b_1 n + b_0)\sigma^n = q(n)\sigma^n$,
  where $b_\imath$ and σ are real numbers and degree of $q(n)$ is $r \in \mathbb{N}_0$

- $(b_r n^r + \ldots + b_1 n + b_0)\sigma^n$ is a sum of geometric forcing functions
- Let $\chi_A(x)$ be the characteristic equation; define

  $$\mu = \begin{cases} 0 & \sigma \text{ is not a root of } \chi_A(x) \\ m & \sigma \text{ is a root of multiplicity } m \text{ of } \chi_A(x) \end{cases}$$

- Then $V_n = n^\mu (d_r n^r + d_{r-1} n^{r-1} + \ldots + d_1 n + d_0)\sigma^n$ may be determined as a particular solution (PS) of LNHR eqn 1.1, $d_\imath$ are complex (or real) constants which have to be determined

- Substitute $V_\imath$ for $T_\imath$ in 1.1 and cancel out the highest power of σ
- In the resulting polynomial equation in n, equate the coefficients of the various powers of n to get a system of linear equations to zero determine the unknown constants $d_\imath$

LNHR for strings with consecutive vowels

Example (Solving LNHR via PS for strings with consecutive vowels)

- $T_n = 21T_{n-1} + 105T_{n-2} + 25 \cdot 26^{n-2}$, $T_0 = 0$, $T_1 = 0$
- $\chi(x) = x^2 - 21x - 105 = 0$, $\{\rho\} = \left\{\frac{1}{2}\left(21 \pm \sqrt{861}\right)\right\}$
- $g(n) = 25 \cdot 26^{n-2}$, comparing with $g(n) = (b_r n^r + b_{r-1} n^{r-1} + \ldots + b_1 n + b_0)\sigma^m$, $\sigma = 26$, $\sigma \notin \{\rho\}$, $V_n = d \cdot 26^n$ (other factors gets absorbed in d)
- Substituting V_n in T_n, $d \cdot 26^n = 21 \cdot d \cdot 26^{n-1} + 105 \cdot d \cdot 26^{n-2} + 25 \cdot 26^{n-2}$
- Factoring out 26^{n-2}, $d \cdot 26^2 = 21d \cdot 26 + 105d + 25$, so d=1
- With $d = 1$, so $V_n = 26^n$
- $T_n = W_n + 26^n$, $W_n = c_1 \left(\frac{21+\sqrt{861}}{2}\right)^n + c_2 \left(\frac{21-\sqrt{861}}{2}\right)^n$

LNHR solution for strings with consecutive vowels (contd.)

Example (Solving LNHR via PS for strings with consecutive vowels (contd.))

- Obtaining initial conditions for W_n as $\{W_\imath = T_\imath - g(\imath)\} = \{W_0 = -1, W_1 = -26\}$
- $c_1 + c_2 = -1,\ c_1\left(\frac{21+\sqrt{861}}{2}\right) + c_2\left(\frac{21-\sqrt{861}}{2}\right) = -26$
- $c_1 = -\left(\frac{31+\sqrt{861}}{2\sqrt{861}}\right),\ c_2 = \left(\frac{31-\sqrt{861}}{2\sqrt{861}}\right)$
- $T_n = 26^n - \frac{31+\sqrt{861}}{2\sqrt{861}}\left(\frac{21+\sqrt{861}}{2}\right)^n - \frac{-31+\sqrt{861}}{2\sqrt{861}}\left(\frac{21-\sqrt{861}}{2}\right)^n$

LNHR where σ is a root of $\chi(x)$

Example ($g(n)$ with σ as root of $\chi(x)$)

- $T_n = T_{n-2} + (n^2 + 1), n \geq 2$, $T_0 = 0$, $T_1 = 1$

LNHR where σ is a root of $\chi(x)$

Example ($g(n)$ with σ as root of $\chi(x)$)

- $T_n = T_{n-2} + (n^2 + 1), n \geq 2$, $T_0 = 0$, $T_1 = 1$
- $\sigma = 1$, $g(n) = (n^2 + 1)1^n$, $\chi(x) = x^2 - 1 = 0$, $\{\rho\} = \{1, -1\}$

LNHR where σ is a root of $\chi(x)$

Example ($g(n)$ with σ as root of $\chi(x)$)

- $T_n = T_{n-2} + (n^2+1), n \geq 2$, $T_0 = 0$, $T_1 = 1$
- $\sigma = 1$, $g(n) = (n^2+1)1^n$, $\chi(x) = x^2 - 1 = 0$, $\{\rho\} = \{1, -1\}$
- $V_n = n^1(d_2 n^2 + d_1 n + d_0)$, as σ is a root of $\chi(x)$
- Substituting, $V_\imath$ for $T_\imath$ in the recurrence equation
- $n\left(d_2 n^2 + d_1 n + d_0\right) =$
  $(n-2)\left(d_2(n-2)^2 + d_1(n-2) + d_0\right) + (n^2+1)$

$\Rightarrow d_2 n^3 + d_1 n^2 + d_0 n =$
$(n-2)\left(d_2\left(n^2 - 4n + 4\right) + d_1(n-2) + d_0\right) + (n^2+1)$

$\Rightarrow d_2 n^3 + d_1 n^2 + d_0 n = d_2 n^3 - 4d_2 n^2 + 4d_2 n - 2d_2 n^2 + 8d_2 n - 8d_2 +$
$d_1 n^2 - 2d_1 n - 2d_1 n + 4d_1 + d_0 n - 2d_0 + (n^2+1)$

$\Rightarrow n^2(d_1 + 4d_2 + 2d_2 - d_1 - 1) +$
$n(d_0 - 4d_2 - 8d_2 + 2d_1 + 2d_1 - d_0) = (-8d_2 + 4d_1 - 2d_0 + 1)$

$$\Rightarrow n^2 \underbrace{(6d_2 - 1)}_{0} + n \underbrace{(-12d_2 + 4d_1)}_{0} = \underbrace{(-8d_2 + 4d_1 - 2d_0 + 1)}_{0}$$

LNHR where σ is a root of $\chi(x)$ (contd.)

Example ($g(n)$ with σ as root of $\chi(x)$ (contd.))

- $6d_2 = 1$, $-12d_2 + 4d_1 = 0$, $8d_2 - 4d_1 + 2d_0 = 1$, so

LNHR where σ is a root of $\chi(x)$ (contd.)

Example ($g(n)$ with σ as root of $\chi(x)$ (contd.))

- $6d_2 = 1$, $-12d_2 + 4d_1 = 0$, $8d_2 - 4d_1 + 2d_0 = 1$, so
- $d_2 = \frac{1}{6}$, $d_1 = \frac{1}{2}$, $d_0 = \frac{5}{6}$, $V_n = \frac{1}{6}n(n^2 + 3n + 5)$,

$$T_n = \underbrace{c1^n + c'(-1)^n}_{W_n} + \underbrace{\frac{1}{6}n(n^2 + 3n + 5)}_{V_n} \; \forall n \in \mathbb{N}_0$$

LNHR where σ is a root of $\chi(x)$ (contd.)

Example ($g(n)$ with σ as root of $\chi(x)$ (contd.))

- $6d_2 = 1$, $-12d_2 + 4d_1 = 0$, $8d_2 - 4d_1 + 2d_0 = 1$, so
- $d_2 = \frac{1}{6}$, $d_1 = \frac{1}{2}$, $d_0 = \frac{5}{6}$, $V_n = \frac{1}{6}n(n^2 + 3n + 5)$,

  $$T_n = \underbrace{c1^n + c'(-1)^n}_{W_n} + \underbrace{\frac{1}{6}n(n^2 + 3n + 5)}_{V_n} \; \forall n \in \mathbb{N}_0$$
- $T_0 = 0$, $T_1 = 1 \Rightarrow c = -\frac{1}{4}, c' = \frac{1}{4}$
- $T_n = -\frac{1}{4} + \frac{1}{4}(-1)^n + \frac{1}{6}n(n^2 + 3n + 5) = \frac{1}{12}\left[2n^3 + 6n^2 + 10n - 3 + 3(-1)^n\right]$

Example of LHNR (roots with multiplicity 2)

Example (LNHR with repeated roots in χ)

- $T_n = 6T_{n-2} - 9T_n + 5 \cdot 3^n, n \geq 2,\ T_0 = 1,\ T_1 = 2$

Example of LHNR (roots with multiplicity 2)

Example (LNHR with repeated roots in χ)

- $T_n = 6T_{n-2} - 9T_n + 5 \cdot 3^n, n \geq 2$, $T_0 = 1$, $T_1 = 2$
- $\sigma = 3$, $g(n) = 5(3^n)$, $\chi(x) = x^2 - 6x + 9 = (x-3)^2 = 0$, $\{\rho\} = \{3, 3\}$

Example of LHNR (roots with multiplicity 2)

Example (LNHR with repeated roots in χ)

- $T_n = 6T_{n-2} - 9T_n + 5 \cdot 3^n, n \geq 2,\ T_0 = 1,\ T_1 = 2$
- $\sigma = 3,\ g(n) = 5(3^n),\ \chi(x) = x^2 - 6x + 9 = (x-3)^2 = 0,$ $\{\rho\} = \{3, 3\}$
- $\mu = 2,\ V_n = dn^2 3^n$, substituting,
- $5 \cdot 3^n = V_n - 6V_{n-1} + 9V_n$, so

Example of LHNR (roots with multiplicity 2)

Example (LNHR with repeated roots in χ)

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- $\sigma = 3,\ g(n) = 5(3^n),\ \chi(x) = x^2 - 6x + 9 = (x-3)^2 = 0,$ $\{\rho\} = \{3, 3\}$
- $\mu = 2,\ V_n = dn^2 3^n$, substituting,
- $5 \cdot 3^n = V_n - 6V_{n-1} + 9V_n$, so
- $d_0 = \frac{5}{18},\ V_n = \frac{5}{18} n^2 3^n,\ T_n = \underbrace{(c_1 + c_2 n)3^n}_{W_n} + \underbrace{\frac{5}{18} n^2 3^n}_{V_n}, \forall n \in \mathbb{N}_0$
- $T_0 = 1,\ T_1 = 2 \Rightarrow c_1 = 1, c_2 = -\frac{11}{18}$
- $T_n = (1 - \frac{11}{18} n + \frac{5}{18} n^2)3^n$

PS of LNHR, $g(n) = \sum_{\jmath} p_j(n)\sigma_{\jmath}^n$

- Let the non-homogeneous term be of the form $p_1(n)\sigma_1^n + p_2(n)\sigma_2^n + \ldots + p_m(n)\sigma_m^n$, $\sigma_\imath$ are mutually distinct
- There is a PS of the form $V_n = V_{1,n} + V_{2,n} + \ldots + V_{m,n}$
  $V_{i,n}$ is a PS of the LNHR
  $T_n = a_1 T_{n-1} + a_2 T_{n-2} + \ldots + a_k T_{n-k} + p_\imath(n)s^n$

LNHR where $g(n) = g_1(n) + g_2(n)$

Example

- $T_n = 2T_{n-1} + 2^n - n$ for $n \geq 1$, $T_0 = 1$ $g(n) = g_1(n) + g_2(n)$ where $g1(n) = 2^n$ and $g_2(n) = -n(1^n)$

LNHR where $g(n) = g_1(n) + g_2(n)$

Example

- $T_n = 2T_{n-1} + 2^n - n$ for $n \geq 1$, $T_0 = 1$ $g(n) = g_1(n) + g_2(n)$ where $g1(n) = 2^n$ and $g_2(n) = -n(1^n)$
- $\chi(x) = x - 2 = 0$, $\{\rho\} = \{2\}$ $V_{1,n}$ for $T_n = 2T_{n-1} + 2^n$ is of the form $V_{1,n} = nd2^n$

LNHR where $g(n) = g_1(n) + g_2(n)$

Example

- $T_n = 2T_{n-1} + 2^n - n$ for $n \geq 1$, $T_0 = 1$ $g(n) = g_1(n) + g_2(n)$ where $g1(n) = 2^n$ and $g_2(n) = -n(1^n)$
- $\chi(x) = x - 2 = 0$, $\{\rho\} = \{2\}$ $V_{1,n}$ for $T_n = 2T_{n-1} + 2^n$ is of the form $V_{1,n} = nd2^n$
- Substituting, $nd2^n = 2(n-1)d2^{n-1} + 2^n$, so $d = 1$, $V_{1,n} = n2^n$ $V_{2,n}$ for $T_n = 2T_{n-1} - n$ is of the form $V - 2, n = dn + d'$

LNHR where $g(n) = g_1(n) + g_2(n)$

Example

- $T_n = 2T_{n-1} + 2^n - n$ for $n \geq 1$, $T_0 = 1$ $g(n) = g_1(n) + g_2(n)$ where $g1(n) = 2^n$ and $g_2(n) = -n(1^n)$
- $\chi(x) = x - 2 = 0$, $\{\rho\} = \{2\}$ $V_{1,n}$ for $T_n = 2T_{n-1} + 2^n$ is of the form $V_{1,n} = nd2^n$
- Substituting, $nd2^n = 2(n-1)d2^{n-1} + 2^n$, so $d = 1$, $V_{1,n} = n2^n$ $V_{2,n}$ for $T_n = 2T_{n-1} - n$ is of the form $V - 2, n = dn + d'$
- Substituting, $dn + d' = 2(d(n-1) + d') - n$, so $d = 1$ and $d' = 2$, $V_{2,n} = n + 2$

LNHR where $g(n) = g_1(n) + g_2(n)$

Example

- $T_n = 2T_{n-1} + 2^n - n$ for $n \geq 1$, $T_0 = 1$ $g(n) = g_1(n) + g_2(n)$ where $g1(n) = 2^n$ and $g_2(n) = -n(1^n)$
- $\chi(x) = x - 2 = 0$, $\{\rho\} = \{2\}$ $V_{1,n}$ for $T_n = 2T_{n-1} + 2^n$ is of the form $V_{1,n} = nd2^n$
- Substituting, $nd2^n = 2(n-1)d2^{n-1} + 2^n$, so $d = 1$, $V_{1,n} = n2^n$ $V_{2,n}$ for $T_n = 2T_{n-1} - n$ is of the form $V - 2, n = dn + d'$
- Substituting, $dn + d' = 2(d(n-1) + d') - n$, so $d = 1$ and $d' = 2$, $V_{2,n} = n + 2$
- Thus, a general solution of $T_n = 2T_{n-1} + 2^n - n$ is $T_n = c2^n + n2^n + n + 2 \; \forall n \in \mathbb{N}_0$

LNHR where $g(n) = g_1(n) + g_2(n)$

Example

- $T_n = 2T_{n-1} + 2^n - n$ for $n \geq 1$, $T_0 = 1$ $g(n) = g_1(n) + g_2(n)$ where $g1(n) = 2^n$ and $g_2(n) = -n(1^n)$
- $\chi(x) = x - 2 = 0$, $\{\rho\} = \{2\}$ $V_{1,n}$ for $T_n = 2T_{n-1} + 2^n$ is of the form $V_{1,n} = nd2^n$
- Substituting, $nd2^n = 2(n-1)d2^{n-1} + 2^n$, so $d = 1$, $V_{1,n} = n2^n$ $V_{2,n}$ for $T_n = 2T_{n-1} - n$ is of the form $V - 2, n = dn + d'$
- Substituting, $dn + d' = 2(d(n-1) + d') - n$, so $d = 1$ and $d' = 2$, $V_{2,n} = n + 2$
- Thus, a general solution of $T_n = 2T_{n-1} + 2^n - n$ is $T_n = c2^n + n2^n + n + 2 \; \forall n \in \mathbb{N}_0$
- $T_0 = 1 = c + 2 \Rightarrow c = -1$, so $T_n = (n-1)2^n + n + 2, \forall n \in \mathbb{N}_0$

Section outline

4. **Deriving solution of LNHR when $g(n) = q(n)\sigma^n$**
   - Binomial identities
   - Convolving binomial coefficients with $g(n)$
   - A PS of LNHR for $g(n) = q(n)\sigma^n$

Binomial identities

Theorem

$$\sum_{r=0}^{n}(-1)^r r^d \binom{n}{r} = 0,\ d < n$$

Corollary

$$\sum_{r=0}^{n}(-1)^r p(r) \binom{n}{r} = 0 \text{ where } p(r) \text{ is a polynomial of degree } d < n$$

- First prove the theorem using the method of induction
- Thereafter, the corollary follows

Binomial identities (contd.)

Induction basis

$$\left.(1-x)^n\right|_{x=1} = \sum_{r=0}^{n}(-1)^r\binom{n}{r} = \sum_{r=0}^{n}(-1)^r r^0\binom{n}{r} = 0$$

Observation

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r} = 1 + \sum_{r=1}^{n}(-1)^r\binom{n}{r} = 1 + \sum_{r=1}^{n}(-1)^r\frac{n}{r}\binom{n-1}{r-1}$$

Induction mechanism

$$\sum_{r=0}^{n}(-1)^r r\binom{n}{r} = 0 + \sum_{r=1}^{n}(-1)^r r\binom{n}{r} =$$

$$\sum_{r=1}^{n}(-1)^r r\frac{n}{r}\binom{n-1}{r-1} = n\sum_{r=1}^{n}(-1)^r\binom{n-1}{r-1} =$$

$$n\sum_{r=0}^{n-1}(-1)^r\binom{n-1}{r} = n\left.(1-x)^{n-1}\right|_{x=1} = 0$$

Binomial identities (contd.)

Induction hypothesis $\displaystyle\sum_{r=0}^{n-1}(-1)^r r^d \binom{n-1}{r} = 0$

Induction step $\displaystyle\sum_{r=0}^{n}(-1)^r r^d \binom{n}{r} = 0 + \sum_{r=1}^{n}(-1)^r r^d \binom{n}{r} =$

$$\sum_{r=1}^{n}(-1)^r r^d \frac{n}{r}\binom{n-1}{r-1} = n\sum_{r=1}^{n}(-1)^r r^{d-1}\binom{n-1}{r-1} = 0$$

Corollary $\displaystyle\sum_{r=0}^{n}(-1)^r p(r)\binom{n}{r} = \sum_{r=0}^{n}(-1)^r \left(\sum_{j=0}^{j=d<n} b_j r^j\right)\binom{n}{r} =$

$$\sum_{j=0}^{j=d<n} b_j \sum_{r=0}^{n}(-1)^r (r^j)\binom{n}{r} = 0$$

Convolving binomial coefficients with $g(n)$

- Let $q(n)$ be a polynomial of degree d, so $p(k) = q(n-k)$ is a polynomial of degree d in k
- $$\sum_{k=0}^{r+1} \binom{r+1}{k} (-1)^k \underbrace{q(n-k)}_{p(k)} = 0, \forall n \qquad (2.1)$$
- Multiplying eq 2.1 by σ^n, we get:

$$\sum_{k=0}^{r+1} \binom{r+1}{k} (-1)^k \underbrace{q(n-k)}_{p(k)} \sigma^n = 0, \forall n \qquad (2.2)$$

Convolving binomial coefficients with $g(n)$ (contd.)

- This may be rewritten as:

$$\sum_{k=0}^{r+1} \underbrace{\binom{r+1}{k} \overbrace{(-1)^k(\sigma)^k}^{(-\sigma)^k}}_{\ell_k} \underbrace{q(n-k)\sigma^{n-k}}_{g(n-k)} = 0, \forall n \qquad (2.3)$$

- And more compactly as:
$\ell_{r+1}q(n)\sigma^n + \ell_r q(n-1)\sigma^{n-1} + \ldots + \ell_0 q(n-(r+1))\sigma^{n-(r+1)} = 0$
- With $g(k) = q(k)\sigma^k$, it may be written as:

$$\ell_{r+1}g(n) + \ell_r g(n-1) + \ldots + \ell_0 g(n-(r+1)) = 0 \qquad (2.4)$$

A PS of LNHR for $g(n) = q(n)\sigma^n$

- Identity (eq 2.4) may be used to eliminate $g(n)$ from (eq 1.1)

$$
\begin{array}{rl}
\ell_{r+1} T_n - \ell_{r+1} a_1 T_{n-1} - \ldots - \ell_{r+1} a_k T_{n-k} = & \ell_{r+1} g(n) \\
\ell_r T_{n-1} - \ell_r a_1 T_{n-2} - \ldots - \ell_r a_k T_{n-k-1} = & \ell_r g(n-1) \\
& \vdots \\
\ell_0 T_{n-(r+1)} - \ell_0 a_1 T_{n-(r+2)} - \ldots - \ell_0 a_k T_{n-(r+k+1)} = & \ell_0 g(n-(r+1)) \\
\hline
\alpha_0 T_n + \alpha_1 T_{n-1} + \ldots + \alpha_{(r+k+1)} T_{n-(r+k+1)} = & 0
\end{array} \tag{3.1}
$$

- Row-wise, replace $T_\jmath$ by $x^\jmath$ and factor out common powers of x and add: $x^{n-k}\ell_{r+1}\chi_A(x) + x^{n-k-1}\ell_r\chi_A(x) + \ldots + x^{n-k-r-1}\ell_0\chi_A(x) = x^{n-k-r-1}\chi_A(x)\left[\ell_{r+1}x^{r+1} + x^r\ell_r + \ldots + \ell_0\right] = \chi_A(x)(x-\sigma)^{r+1}x^{n-k-r-1} = \ell_{r+1}g(n) + \ell_r g(n-1) + \ldots + \ell_0 g(n-(r+1)) = 0$
- Characteristic equation of eq 3.1: $\chi_\alpha(x) = \chi_A(x)(x-\sigma)^{r+1} = 0$

A PS of LNHR for $g(n) = q(n)\sigma^n$ (contd)

- We have $\chi_\alpha(x) = \chi_A(x)(x - \sigma)^{r+1} = 0$
- If σ is not a root of $\chi_A(x)$, eq 3.1 will have solutions

A PS of LNHR for $g(n) = q(n)\sigma^n$ (contd)

- We have $\chi_\alpha(x) = \chi_A(x)(x-\sigma)^{r+1} = 0$
- If σ is not a root of $\chi_A(x)$, eq 3.1 will have solutions $\sum_{s=0}^{s=r}(d_s n^s)\,\sigma^n$ which may be considered PS $V_n = p(n)\sigma^n$ of eq 1.1

A PS of LNHR for $g(n) = q(n)\sigma^n$ (contd)

- We have $\chi_\alpha(x) = \chi_A(x)(x - \sigma)^{r+1} = 0$
- If σ is not a root of $\chi_A(x)$, eq 3.1 will have solutions $\sum_{s=0}^{s=r} (d_s n^s)\, \sigma^n$ which may be considered PS $V_n = p(n)\sigma^n$ of eq 1.1
- If σ is a root of $\chi_A(x)$ of multiplicity m, eq 3.1 will have solutions

A PS of LNHR for $g(n) = q(n)\sigma^n$ (contd)

- We have $\chi_\alpha(x) = \chi_A(x)(x-\sigma)^{r+1} = 0$
- If σ is not a root of $\chi_A(x)$, eq 3.1 will have solutions $\sum_{s=0}^{s=r} (d_s n^s)\, \sigma^n$ which may be considered PS $V_n = p(n)\sigma^n$ of eq 1.1
- If σ is a root of $\chi_A(x)$ of multiplicity m, eq 3.1 will have solutions $\sum_{s=0}^{s=r+m} (d'_s n^s)\, \sigma^n$
- From these, PS of eq 1.1 may be considered as

$V_n =$

A PS of LNHR for $g(n) = q(n)\sigma^n$ (contd)

- We have $\chi_\alpha(x) = \chi_A(x)(x-\sigma)^{r+1} = 0$
- If σ is not a root of $\chi_A(x)$, eq 3.1 will have solutions $\sum_{s=0}^{s=r} (d_s n^s)\,\sigma^n$ which may be considered PS $V_n = p(n)\sigma^n$ of eq 1.1
- If σ is a root of $\chi_A(x)$ of multiplicity m, eq 3.1 will have solutions $\sum_{s=0}^{s=r+m} (d'_s n^s)\,\sigma^n$
- From these, PS of eq 1.1 may be considered as $V_n = \sum_{s=m}^{s=r+m} (d'_s n^s)\,\sigma^n = n^s p(n)\sigma^n$

A PS of LNHR for $g(n) = q(n)\sigma^n$ (contd)

- We have $\chi_\alpha(x) = \chi_A(x)(x-\sigma)^{r+1} = 0$
- If σ is not a root of $\chi_A(x)$, eq 3.1 will have solutions $\sum_{s=0}^{s=r} (d_s n^s)\,\sigma^n$ which may be considered PS $V_n = p(n)\sigma^n$ of eq 1.1
- If σ is a root of $\chi_A(x)$ of multiplicity m, eq 3.1 will have solutions $\sum_{s=0}^{s=r+m} (d'_s n^s)\,\sigma^n$
- From these, PS of eq 1.1 may be considered as
  $V_n = \sum_{s=m}^{s=r+m} (d'_s n^s)\,\sigma^n = n^s p(n)\sigma^n$
- Both cases are covered via
  $V_n = n^\mu(d_r n^r + d_{r-1}n^{r-1} + \ldots + d_1 n + d_0)\sigma^n$,
  $\mu = \begin{cases} 0 & \sigma \text{ is not a root of } \chi_A(x) \\ m & \sigma \text{ is a root of multiplicity } m \text{ of } \chi_A(x) \end{cases}$

Section outline

5. **Divide and conquer recurrences**
   - Revisiting $T_n = 2T_{\frac{n}{2}} + an$
   - DC result for $T_n = aT_{\frac{n}{s}} + g(n)$
   - Example of the form $T_n = aT_{\frac{n}{s}} + p(n)$
   - DC recurrence when $a = s^r$, $n = s^m$, $m \geq 0$
   - DC recurrence when $a \neq s^r$, $n = s^m$, $m \geq 0$
   - Alternate statement of DC recurrence

Revisiting $T_n = 2T_{\frac{n}{2}} + an$

Example ($T_n = 2T_{\frac{n}{2}} + an$, $T_1 = 1$)

- It's a linear non-homogeneous recurrence with constant coeefficients, but not of constant order

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- It's a linear non-homogeneous recurrence with constant coeefficients, but not of constant order
- Let $n = 2^m$, $T_n = T_{2^m} = S_m$ – we actually skip over many intermediate sizes
- $S_m = 2S_{m-1} + a2^m$, $S_0 = 1$ – LNHR
- $\chi_A(x) = x - 2$, $\{\rho\} = \{2\}$
- $V_m = md2^m$, substituting, $md2^m = 2(m-1)d2^{m-1} + a2^m$
  $md = (m-1)d + a \Rightarrow d = a$, $V_m = am2^m$
- $S_m = c2^m + am2^m$, $S_0 = 1 \Rightarrow c = 1$, so $S_m = \underbrace{2^m}_{n}(a\underbrace{m}_{\lg n}+1)$
- Finally, $S_m = T_{2^m} = T_n = n(a \lg n + 1) \in \Theta(n \lg n)$, for $n = 2^m$, $m \geq 0$

DC result for $T_n = aT_{\frac{n}{s}} + g(n)$

When $n = s^m$, $T_n = aT_{\frac{n}{s}} + g(n)$ can be solved with the transformation $n = s^m$

Theorem ($T_n = aT_{\frac{n}{s}} + g(n)$ for $n = s^m$)

- *Let $g(n)$ (degree $r \in \mathbb{N}_0$) be* $g(n) = b_r n^r + b_{r-1} n^{r-1} + \ldots + b_1 n + b_0$, $b_0, b_1, \ldots, b_t \in \mathbb{R}, b_r > 0$
- *Let T_n be a monotonically increasing sequence that satisfies $T_n = aT_{\frac{n}{s}} + g(n)$ whenever $n = s^m$, then ($\varepsilon > 0$ below)*
- $T_n \in \begin{cases} \Theta(n^r) & a < s^r \quad [\log_s a + \varepsilon = r \Rightarrow g(n) \in \Omega\left(n^{\log_s a}\right)] \\ \Theta(n^r \log n) & a = s^r \quad [\log_s a = r \Rightarrow g(n) \in \Theta\left(n^{\log_s a}\right)] \\ \Theta(n^{\log_s a}) & a > s^r \quad [\log_s a - \varepsilon = r \Rightarrow g(n) \in \mathcal{O}\left(n^{\log_s a}\right)] \end{cases}$

Examples of DC recurrence of the form $T_n = aT_{\frac{n}{s}} + p(n)$, $p(n)$ is polynomial of degree r

$$T_n \in \begin{cases} \Theta(n^r) & a < s^r \\ \Theta(n^r \log_s n) & a = s^r \\ \Theta(n^{\log_s a}) & a > s^r \end{cases} , n = s^m, m \geq 0$$

Example

- Binary search

  $T_n = T_{\frac{n}{2}} + c; [p(n) = c]$

  $a = 1, s = 2, r = 0,\ s^r = 1,\ a = s^r$

  $T_n \in \Theta(n^r \lg n) = \Theta(\lg n)$

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- Strange recurrence

  $T_n = T_{\frac{n}{2}} + bn + c; [p(n) = bn + c]$

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Examples of DC recurrence of the form $T_n = aT_{\frac{n}{s}} + p(n)$, $p(n)$ is polynomial of degree r

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  $a = 1, s = 2, r = 1$, $s^r = 2$, $a < s^r$

  $T_n \in \Theta(n^r) = \Theta(n)$

- FFT

  $T_n = 2T_{\frac{n}{2}} + bn + c$; $[p(n) = bn + c]$

  $a = 2, s = 2, r = 1$, $s^r = 2$, $a = s^r$

  $T_n \in \Theta(n^r \lg n) = \Theta(n \lg n)$

Examples of DC recurrence of the form $T_n = aT_{\frac{n}{s}} + p(n)$, $p(n)$ is polynomial of degree r

$$T_n \in \begin{cases} \Theta(n^r) & a < s^r \\ \Theta(n^r \log_s n) & a = s^r \\ \Theta(n^{\log_s a}) & a > s^r \end{cases}, n = s^m, m \geq 0$$

Example

- Binary search

  $T_n = T_{\frac{n}{2}} + c; [p(n) = c]$

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- Strange recurrence

  $T_n = T_{\frac{n}{2}} + bn + c; [p(n) = bn + c]$

  $a = 1, s = 2, r = 1, s^r = 2, a < s^r$

  $T_n \in \Theta(n^r) = \Theta(n)$

- FFT

  $T_n = 2T_{\frac{n}{2}} + bn + c; [p(n) = bn + c]$

  $a = 2, s = 2, r = 1, s^r = 2, a = s^r$

  $T_n \in \Theta(n^r \lg n) = \Theta(n \lg n)$

- Karatsuba multiplication

  $T_n = 3T_{\frac{n}{2}} + bn + c; [p(n) = bn + c]$

  $a = 3, s = 2, r = 1, s^r = 2, a > s^r$

  $T_n \in \Theta(n^{\log_s a}) = \Theta(n^{\log_2 3})$, $\log_2 3 = 1.58496\ldots$

DC recurrence when $a = s^r$, $n = s^m$, $m \geq 0$

Proof for DC recurrence when $a = s^r$.

- $f(n) = af(\frac{n}{s}) + bn^r$

DC recurrence when $a = s^r$, $n = s^m$, $m \geq 0$

Proof for DC recurrence when $a = s^r$.

- $f(n) = af(\frac{n}{s}) + bn^r$

$$= a\left(af\left(\frac{n}{s^2}\right) + b\left(\frac{n}{s}\right)^r\right) + bn^r = a^2 f\left(\frac{n}{s^2}\right) + ab\left(\frac{n}{s}\right)^r + bn^r$$

DC recurrence when $a = s^r$, $n = s^m$, $m \geq 0$

Proof for DC recurrence when $a = s^r$.

- $$\begin{aligned} f(n) &= af(\tfrac{n}{s}) + bn^r \\ &= a\left(af\left(\tfrac{n}{s^2}\right) + b\left(\tfrac{n}{s}\right)^r\right) + bn^r = a^2 f\left(\tfrac{n}{s^2}\right) + ab\left(\tfrac{n}{s}\right)^r + bn^r \\ &= a^2\left(af\left(\tfrac{n}{s^3}\right) + b\left(\tfrac{n}{s^2}\right)^r\right) + ab\left(\tfrac{n}{s}\right)^r + bn^r \\ &= a^3 f\left(\tfrac{n}{s^3}\right) + a^2 b\left(\tfrac{n}{s^2}\right)^r + ab\left(\tfrac{n}{s}\right)^r + bn^r \\ &= \ldots = a^m f(1) + \sum_{j=0}^{m-1} a^j b\left(\tfrac{n}{s^j}\right)^r \end{aligned}$$

DC recurrence when $a = s^r$, $n = s^m$, $m \geq 0$

Proof for DC recurrence when $a = s^r$.

- $$f(n) = af(\tfrac{n}{s}) + bn^r$$
$$= a\left(af\left(\tfrac{n}{s^2}\right) + b(\tfrac{n}{s})^r\right) + bn^r = a^2 f\left(\tfrac{n}{s^2}\right) + ab\left(\tfrac{n}{s}\right)^r + bn^r$$
$$= a^2\left(af\left(\tfrac{n}{s^3}\right) + b(\tfrac{n}{s^2})^r\right) + ab\left(\tfrac{n}{s}\right)^r + bn^r$$
$$= a^3 f\left(\tfrac{n}{s^3}\right) + a^2 b(\tfrac{n}{s^2})^r + ab\left(\tfrac{n}{s}\right)^r + bn^r$$
$$= \ldots = a^m f(1) + \sum_{j=0}^{m-1} a^j b(\tfrac{n}{s^j})^r = a^m f(1) + \sum_{j=0}^{m-1} s^{rj} b(\tfrac{n}{s^j})^r$$

DC recurrence when $a = s^r$, $n = s^m$, $m \geq 0$

Proof for DC recurrence when $a = s^r$.

- $$\begin{aligned} f(n) &= af(\tfrac{n}{s}) + bn^r \\ &= a\left(af\left(\tfrac{n}{s^2}\right) + b(\tfrac{n}{s})^r\right) + bn^r = a^2 f\left(\tfrac{n}{s^2}\right) + ab\left(\tfrac{n}{s}\right)^r + bn^r \\ &= a^2\left(af\left(\tfrac{n}{s^3}\right) + b(\tfrac{n}{s^2})^r\right) + ab\left(\tfrac{n}{s}\right)^r + bn^r \\ &= a^3 f\left(\tfrac{n}{s^3}\right) + a^2 b(\tfrac{n}{s^2})^r + ab\left(\tfrac{n}{s}\right)^r + bn^r \\ &= \ldots = a^m f(1) + \sum_{j=0}^{m-1} a^j b(\tfrac{n}{s^j})^r = a^m f(1) + \sum_{j=0}^{m-1} s^{rj} b(\tfrac{n}{s^j})^r \\ &= a^m f(1) + \sum_{j=0}^{m-1} bn^r = a^m f(1) + bmn^r = a^m f(1) + b\left(\log_s n\right) n^r \end{aligned}$$

DC recurrence when $a = s^r$, $n = s^m$, $m \geq 0$

Proof for DC recurrence when $a = s^r$.

- $$\begin{aligned} f(n) &= af(\tfrac{n}{s}) + bn^r \\ &= a\left(af\left(\tfrac{n}{s^2}\right) + b(\tfrac{n}{s})^r\right) + bn^r = a^2 f\left(\tfrac{n}{s^2}\right) + ab\left(\tfrac{n}{s}\right)^r + bn^r \\ &= a^2\left(af\left(\tfrac{n}{s^3}\right) + b(\tfrac{n}{s^2})^r\right) + ab\left(\tfrac{n}{s}\right)^r + bn^r \\ &= a^3 f\left(\tfrac{n}{s^3}\right) + a^2 b(\tfrac{n}{s^2})^r + ab\left(\tfrac{n}{s}\right)^r + bn^r \\ &= \ldots = a^m f(1) + \sum_{\jmath=0}^{m-1} a^\jmath b(\tfrac{n}{s^\jmath})^r = a^m f(1) + \sum_{\jmath=0}^{m-1} s^{r\jmath} b(\tfrac{n}{s^\jmath})^r \\ &= a^m f(1) + \sum_{\jmath=0}^{m-1} bn^r = a^m f(1) + bmn^r = a^m f(1) + b\left(\log_s n\right) n^r \\ &= s^{rm} f(1) + b\left(\log_s n\right) n^r = (s^m)^r f(1) + b\left(\log_s n\right) n^r \\ &= n^r f(1) + bn^r \log_s n \end{aligned}$$

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Proof for DC recurrence when $a = s^r$.

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- $f(n) \in \Theta(n^r \log_s n)$ if $a = s^r$, $n = s^m$, $m \geq 0$ □

DC recurrence when $a \neq s^r$, $n = s^m$, $m \geq 0$

Proof for DC recurrence when $a \neq s^r$.

- $T(n) = c_1 n^r + c_2 n^{\log_s a}$, $c_1 = s^r \frac{b}{s^r - a}$, $c_2 = T(1) + s^r \frac{b}{a - s^r}$ [Ind hypothesis]

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  Basis: $m = 0$ ($n = s^m = 1$) $T(1) = c_1 n^r + c_2 n^{\log_s a} = c_1 + c_2 = T(1)$ ✓

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  Inductive step: $m > 0$, assume true for $n \leq s^{m-1}$, check for $n = s^m$

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  Basis: $m = 0$ $(n = s^m = 1)$ $T(1) = c_1 n^r + c_2 n^{\log_s a} = c_1 + c_2 = T(1)$ ✓
  Inductive step: $m > 0$, assume true for $n \leq s^{m-1}$, check for $n = s^m$
- $$T(n) = af\left(\frac{n}{s}\right) + bn^r = a\left[\underbrace{\left(s^r \frac{b}{s^r - a}\right)\left(\frac{n}{s}\right)^r + \left(T(1) + s^r \frac{b}{a - s^r}\right)\left(\frac{n}{s}\right)^{\log_s a}}_{\text{by induction hypothesis, } \frac{n}{s} = s^{m-1}}\right] + bn^r =$$

DC recurrence when $a \neq s^r$, $n = s^m$, $m \geq 0$

Proof for DC recurrence when $a \neq s^r$.

- $T(n) = c_1 n^r + c_2 n^{\log_s a}$, $c_1 = s^r \frac{b}{s^r - a}$, $c_2 = T(1) + s^r \frac{b}{a - s^r}$ [Ind hypothesis]
  Basis: $m = 0$ ($n = s^m = 1$) $T(1) = c_1 n^r + c_2 n^{\log_s a} = c_1 + c_2 = T(1)$ ✓
  Inductive step: $m > 0$, assume true for $n \leq s^{m-1}$, check for $n = s^m$
- $T(n) = af(\frac{n}{s}) + bn^r = a \left[\underbrace{\left(s^r \frac{b}{s^r - a}\right)\left(\frac{n}{s}\right)^r + \left(T(1) + s^r \frac{b}{a - s^r}\right)\left(\frac{n}{s}\right)^{\log_s a}}_{\text{by induction hypothesis, } \frac{n}{s} = s^{m-1}} \right] + bn^r =$

  $s^r \left(\frac{ab}{s^r - a}\right) \frac{n^r}{s^r} + a\left(T(1) + s^r \frac{b}{a - s^r}\right) \frac{n^{\log_s a}}{a} + \frac{b(s^r - a)n^r}{(s^r - a)} =$

DC recurrence when $a \neq s^r$, $n = s^m$, $m \geq 0$

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  Basis: $m = 0$ ($n = s^m = 1$) $T(1) = c_1 n^r + c_2 n^{\log_s a} = c_1 + c_2 = T(1)$ ✓
  Inductive step: $m > 0$, assume true for $n \leq s^{m-1}$, check for $n = s^m$
- $T(n) = af(\frac{n}{s}) + bn^r = a \left[\underbrace{\left(s^r \frac{b}{s^r - a}\right)\left(\frac{n}{s}\right)^r + \left(T(1) + s^r \frac{b}{a - s^r}\right)\left(\frac{n}{s}\right)^{\log_s a}}_{\text{by induction hypothesis, } \frac{n}{s} = s^{m-1}} \right] + bn^r =$

  $s^r \left(\frac{ab}{s^r - a}\right) \frac{n^r}{s^r} + a\left(T(1) + s^r \frac{b}{a - s^r}\right) \frac{n^{\log_s a}}{a} + \frac{b(s^r - a)n^r}{(s^r - a)} =$

  $\left(\frac{ab + b(s^r - a)}{s^r - a}\right) n^r + \left(T(1) + s^r \frac{b}{a - s^r}\right) n^{\log_s a} =$

DC recurrence when $a \neq s^r$, $n = s^m$, $m \geq 0$

Proof for DC recurrence when $a \neq s^r$.

- $T(n) = c_1 n^r + c_2 n^{\log_s a}$, $c_1 = s^r \frac{b}{s^r - a}$, $c_2 = T(1) + s^r \frac{b}{a - s^r}$ [Ind hypothesis]
  Basis: $m = 0$ $(n = s^m = 1)$ $T(1) = c_1 n^r + c_2 n^{\log_s a} = c_1 + c_2 = T(1)$ ✓
  Inductive step: $m > 0$, assume true for $n \leq s^{m-1}$, check for $n = s^m$
- $$T(n) = af(\tfrac{n}{s}) + bn^r = a\left[\underbrace{\left(s^r \frac{b}{s^r - a}\right)\left(\frac{n}{s}\right)^r + \left(T(1) + s^r \frac{b}{a - s^r}\right)\left(\frac{n}{s}\right)^{\log_s a}}_{\text{by induction hypothesis, } \frac{n}{s} = s^{m-1}}\right] + bn^r =$$

$$s^r\left(\frac{ab}{s^r - a}\right)\frac{n^r}{s^r} + a\left(T(1) + s^r \frac{b}{a - s^r}\right)\frac{n^{\log_s a}}{a} + \frac{b(s^r - a)n^r}{(s^r - a)} =$$

$$\left(\frac{ab + b(s^r - a)}{s^r - a}\right) n^r + \left(T(1) + s^r \frac{b}{a - s^r}\right) n^{\log_s a} =$$

$$\left(s^r \frac{b}{s^r - a}\right) n^r + \left(T(1) + s^r \frac{b}{a - s^r}\right) n^{\log_s a} = c_1 n^r + c_2 n^{\log_s a} \checkmark$$

DC recurrence when $a \neq s^r$, $n = s^m$, $m \geq 0$

Proof for DC recurrence when $a \neq s^r$.

- $T(n) = c_1 n^r + c_2 n^{\log_s a}$, $c_1 = s^r \frac{b}{s^r - a}$, $c_2 = T(1) + s^r \frac{b}{a - s^r}$ [Ind hypothesis]
  Basis: $m = 0$ ($n = s^m = 1$) $T(1) = c_1 n^r + c_2 n^{\log_s a} = c_1 + c_2 = T(1)$ ✓
  Inductive step: $m > 0$, assume true for $n \leq s^{m-1}$, check for $n = s^m$
- $T(n) = af(\frac{n}{s}) + bn^r = a \left[\underbrace{\left(s^r \frac{b}{s^r - a}\right)\left(\frac{n}{s}\right)^r + \left(T(1) + s^r \frac{b}{a - s^r}\right)\left(\frac{n}{s}\right)^{\log_s a}}_{\text{by induction hypothesis, } \frac{n}{s} = s^{m-1}} \right] + bn^r =$
  $s^r \left(\frac{ab}{s^r - a}\right) \frac{n^r}{s^r} + a\left(T(1) + s^r \frac{b}{a - s^r}\right) \frac{n^{\log_s a}}{a} + \frac{b(s^r - a)n^r}{(s^r - a)} =$
  $\left(\frac{ab + b(s^r - a)}{s^r - a}\right) n^r + \left(T(1) + s^r \frac{b}{a - s^r}\right) n^{\log_s a} =$
  $\left(s^r \frac{b}{s^r - a}\right) n^r + \left(T(1) + s^r \frac{b}{a - s^r}\right) n^{\log_s a} = c_1 n^r + c_2 n^{\log_s a}$ ✓

- If $a < s^r$, then $\log_s a < r$, so $T(n) = c_1 n^r + c_2 n^{\log_s a} \leq (c_1 + c_2) n^r \in \Theta(n^r)$

DC recurrence when $a \neq s^r$, $n = s^m$, $m \geq 0$

Proof for DC recurrence when $a \neq s^r$.

- $T(n) = c_1 n^r + c_2 n^{\log_s a}$, $c_1 = s^r \frac{b}{s^r - a}$, $c_2 = T(1) + s^r \frac{b}{a - s^r}$ [Ind hypothesis]
  Basis: $m = 0$ ($n = s^m = 1$) $T(1) = c_1 n^r + c_2 n^{\log_s a} = c_1 + c_2 = T(1)$ ✓
  Inductive step: $m > 0$, assume true for $n \leq s^{m-1}$, check for $n = s^m$
- $$T(n) = af(\tfrac{n}{s}) + bn^r = a\left[\underbrace{\left(s^r \frac{b}{s^r - a}\right)\left(\frac{n}{s}\right)^r + \left(T(1) + s^r \frac{b}{a - s^r}\right)\left(\frac{n}{s}\right)^{\log_s a}}_{\text{by induction hypothesis, } \frac{n}{s} = s^{m-1}}\right] + bn^r =$$
  $$s^r\left(\frac{ab}{s^r - a}\right)\frac{n^r}{s^r} + a\left(T(1) + s^r \frac{b}{a - s^r}\right)\frac{n^{\log_s a}}{a} + \frac{b(s^r - a)n^r}{(s^r - a)} =$$
  $$\left(\frac{ab + b(s^r - a)}{s^r - a}\right)n^r + \left(T(1) + s^r \frac{b}{a - s^r}\right)n^{\log_s a} =$$
  $$\left(s^r \frac{b}{s^r - a}\right)n^r + \left(T(1) + s^r \frac{b}{a - s^r}\right)n^{\log_s a} = c_1 n^r + c_2 n^{\log_s a} \ ✓$$

- If $a < s^r$, then $\log_s a < r$, so $T(n) = c_1 n^r + c_2 n^{\log_s a} \leq (c_1 + c_2)n^r \in \Theta(n^r)$
- If $a > s^r$, then $\log_s a > r$, so $T(n) = c_1 n^r + c_2 n^{\log_s a} \leq (c_1 + c_2)n^{\log_s a} \in \Theta(n^{\log_s a})$ □

Alternate statement of DC recurrence

Consider the recurrence $T(n) = aT\left(\frac{n}{s}\right) + f(n)$.

- If $f(n) = \mathcal{O}\left(n^{\log_s a - \varepsilon}\right)$ for some constant $\varepsilon > 0$, then $T(n) = \mathcal{O}\left(n^{\log_s a}\right)$
- If $f(n) = \Theta\left(n^{\log_s a}\right)$, then $T(n) = \Theta\left(n^{\log_s a} \log n\right)$
- If $f(n) = \Omega\left(n^{\log_s a + \varepsilon}\right)$ for some constant $\varepsilon > 0$, and if f satisfies the smoothness condition $af\left(\frac{n}{s}\right) \leq cf(n)$ for some constant $c < 1$, then $T(n) = \Theta\left(f(n)\right)$

For detailed proof see: `https://www.cs.cornell.edu/courses/cs3110/2012sp/lectures/lec20-master/mm-proof.pdf` □

Section outline

6 Practice problems

Practice problems

Some quick sort recurrences

- The sorting time of quick sort depends on the placement of the pivot; solve these cases:
  - Pivot is always placed at position k, so $T(n) = T(k) + T(n-k) + an$, $T(k) = b$ (when $n > k$, consider $n \leq k$ to be base cases sorted in constant time using any other sorting procedure)
  - Pivot is always placed to split the array in ratio $\alpha : (1-\alpha)$, so $T(n) = T(\alpha n) + T((1-\alpha)n) + an$, $T(1) = b$

Example ($\alpha = \frac{1}{4}$)

- $T(n) = T\left(\frac{n}{4}\right) + T\left(\frac{3n}{4}\right) + an$, $T(1) = b$
- Depth for the (longer) $\frac{3}{4}$ branch is $\log_{\frac{4}{3}} n$
- Contribution of each level (until shorter branch is exhausted) is $\leq \max(a,b)n$
- The overall contribution for all the levels is $\leq \max(a,b)n\log_{\frac{4}{3}} n \in \mathcal{O}(n \log n)$

- For α, the depth d will satisfy $\alpha^d = \frac{1}{n}$, so, $n = \left(\frac{1}{\alpha}\right)^d \Rightarrow d = \log_{\frac{1}{\alpha}} n$
- $T(n) \leq \underbrace{\max(a,b)n}_{\text{max contribtion at level}} \underbrace{\max\left(\log_{\frac{1}{\alpha}} n, \log_{\frac{1}{(1-\alpha)}} n\right)}_{\text{depth of recursion tree}} \in \Theta(n \log n)$

Practice problems (contd.)

Use the Master Theorem or DC recurrence technique to asymptotically solve each of the following recurrences or state why those doesn't apply.

- $T(n) = 4T\left(\frac{n}{2}\right) + n$
- $T(n) = 4T\left(\frac{n}{2}\right) + n^2$
- $T(n) = 4T\left(\frac{n}{2}\right) + n^3$
- $T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n}$
- $T(n) = 2T\left(\frac{n}{4}\right) + n^{0.51}$
- $T(n) = 8T\left(\frac{n}{3}\right) + n!$
- $T(n) = T(n-1) + \sqrt{\pi}$
- $T(n) = 4T\left(\frac{n}{2}\right) + n^2 + n$
- $T(n) = T\left(\frac{n}{2}\right) + n\left(n\sin\left(n - \frac{\pi}{2}\right) + 2\right)$

NB $n! \in \omega\left(n^n\right)$

NB $n! \in \omega\left(2^n\right)$

NB $\lg n! \in \Theta\left(\left(n \lg n\right)\right)$

Practice problems (contd.)

- $T(n) = 3T\left(\frac{n}{2}\right) + n \lg n$
- $T(n) = 4T\left(\frac{n}{2}\right) + n^2 \lg n$
- $T(n) = 5T\left(\frac{n}{2}\right) + n^2 \lg n$
- $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\lg n}$
- $T(n) = T\left(\frac{n}{4}\right) + \lg n$
- $T(n) = 2T\left(\frac{n}{4}\right) + \lg n$
- $T(n) = T\left(\frac{n}{2}\right) + \lg n!$
- $T(n) = 4T\left(\frac{n}{2}\right) + \frac{n}{\lg \lg n}$
- $T(n) = 2T\left(\sqrt{n}\right) + \lg n$

Practice problems (contd.)

- $$T_n = \begin{cases} 0 & n = 0 \\ 2 & n = 1 \\ T_{n-1} + 2U_{n-1} & n \geq 2 \end{cases}$$
  $$U_n = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ 2T_{n-1} + 3U_{n-1} & n \geq 2 \end{cases}$$

Practice problems (contd.)

Josephus recurrence

- 41 rebels were trapped by the Romans at the Jotapata fortress. Instead of surrendering and facing painful consequences, they made a suicide pact
- They were to stand in a circle, every third man was to be killed, the last man was to kill himself
- Flavius Josephus and a friend wanted to survive
- At what positions would they have to stand to be the last two surviving positions?
- Develop a recurrence J_n^k for the position of the last person alive in a circle of n people where the k^{th} person must fall every time