On Functionality of Quadraginta Octants of Naive Sphere with Application to Circle Drawing

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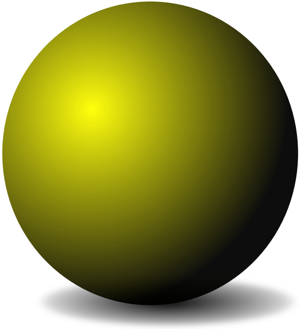
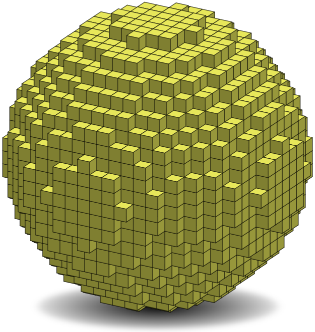
DGCI 2016 Nantes

Definition *(Naive Sphere)*



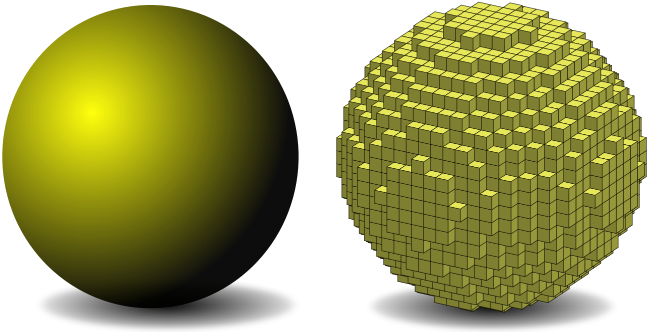
real sphere



naive sphere

Naive sphere = 2-minimal +
each voxel is close to the real sphere as much as possible.

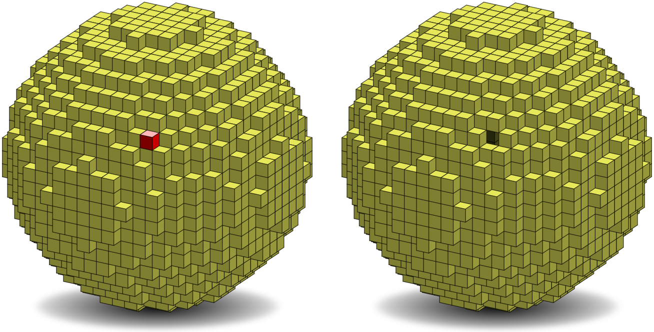
Definition *(Naive Sphere)*



real sphere naive sphere

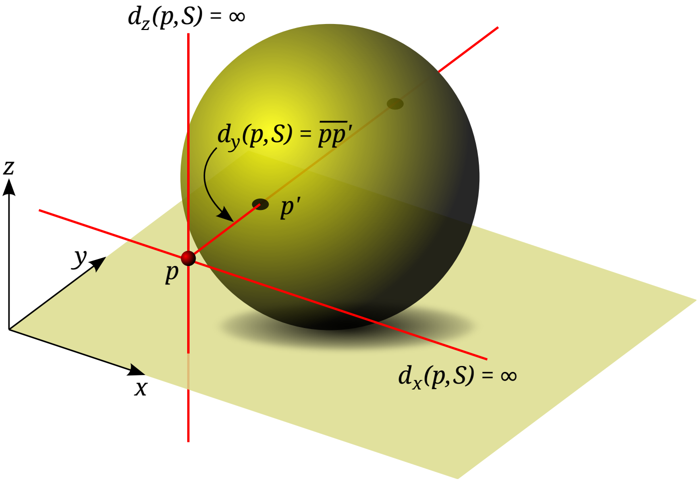
Naive sphere = 2-minimal +
each voxel is close to the real sphere as much as possible.

2-minimal



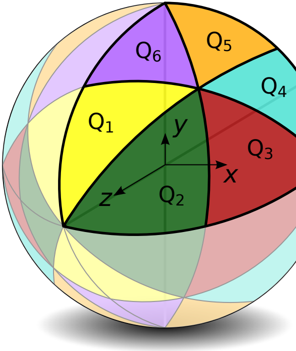
Removal of *any voxel* forms a tunnel.

Isothetic distance

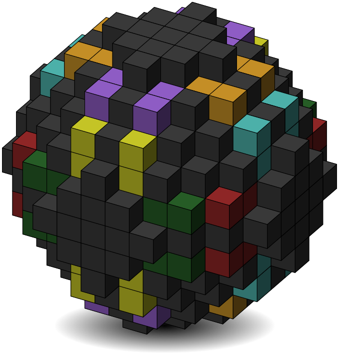


$$d_\perp(p,S) := \min\{d_x(p,S), d_y(p,S), d_z(p,S)\} \leqslant \frac{1}{2}.$$

Symmetry *(quadraginta octants)*

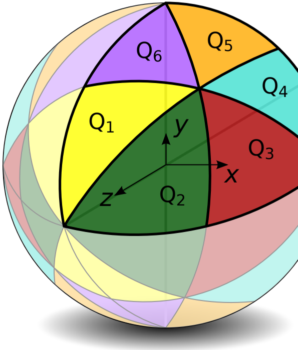


$r = 5$

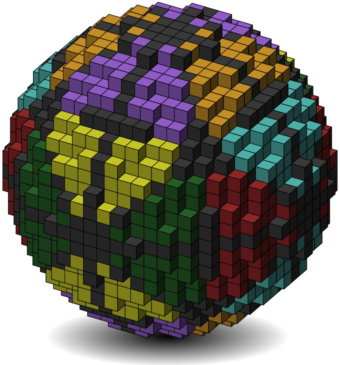


#q-octants = #permutations of $(\pm x, \pm y, \pm z) = 3! \times 2^3 = 48$.

Symmetry *(quadraginta octants)*



$r = 10$



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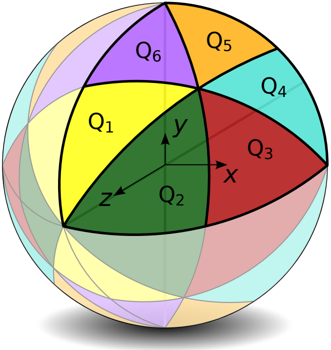
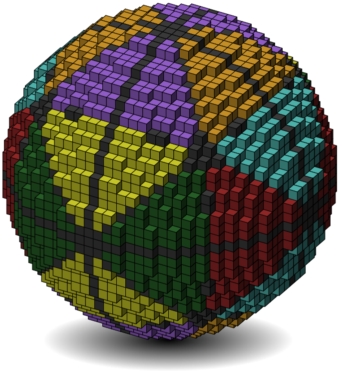


$r = 15$



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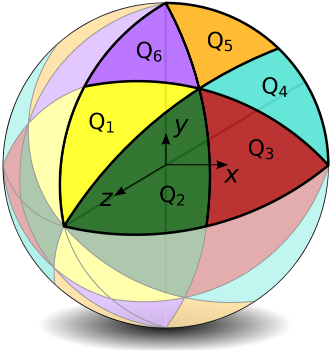

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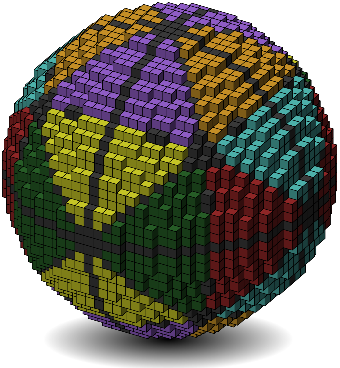

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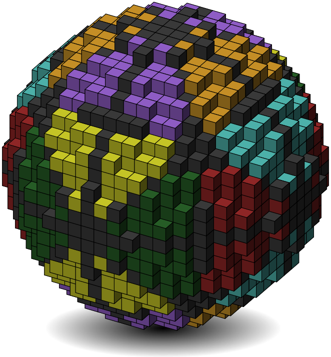
Equation *(Naive Sphere)*

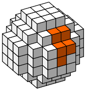


Theorem (Naive Sphere)

$$\mathsf{S}_r = \left\{ p \in \mathbb{Z}^3 : \left(r^2 - \max(X) \leqslant s < r^2 + \max(X)\right) \wedge \left(\left(s \neq r^2 + \max(X) - 1\right) \vee \left(\mathrm{mid}(X) \neq \max(X)\right)\right) \right\}.$$

Here $p = (i, j, k) \in \mathbb{Z}^3, s = i^2 + j^2 + k^2, X = \{|i|, |j|, |k|\}$.

Functional Plane



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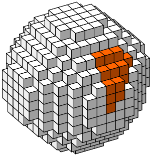

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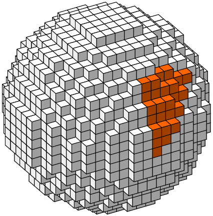
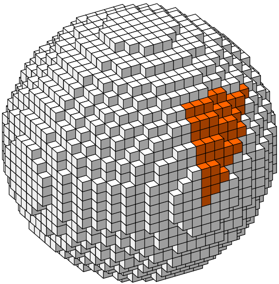
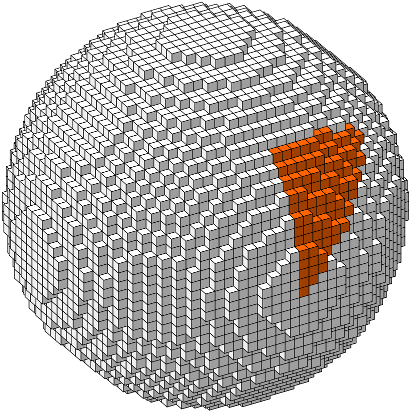


Functional Plane



Functional Plane



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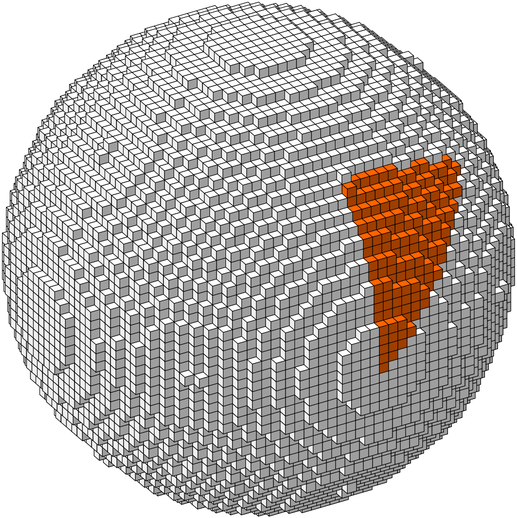
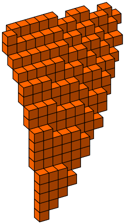


Functional Plane



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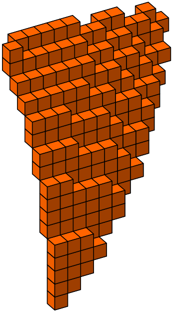

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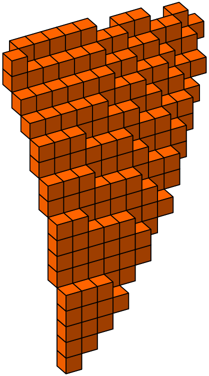

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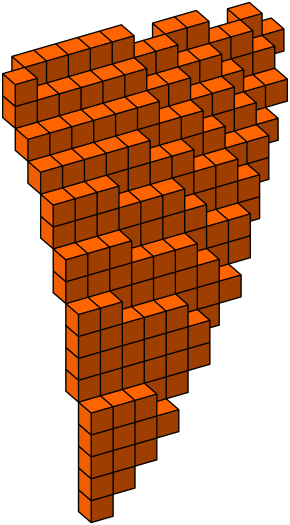

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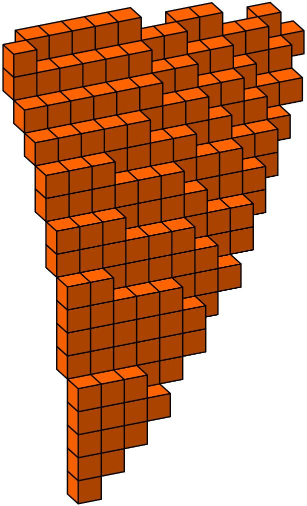

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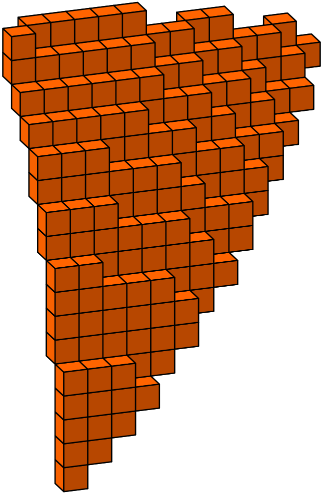

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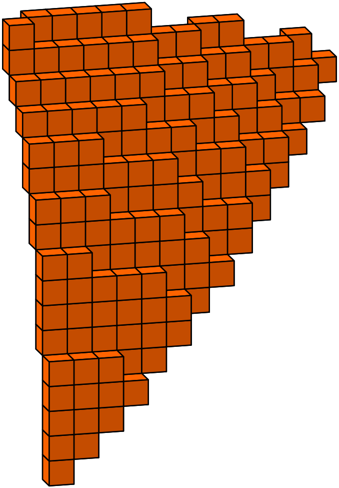

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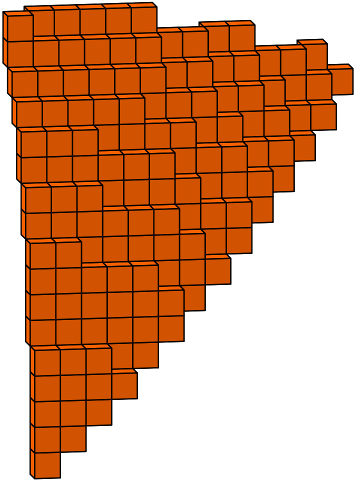

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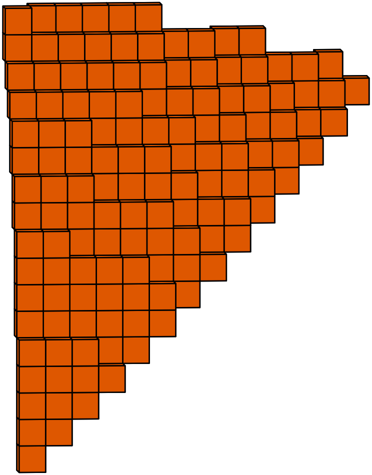

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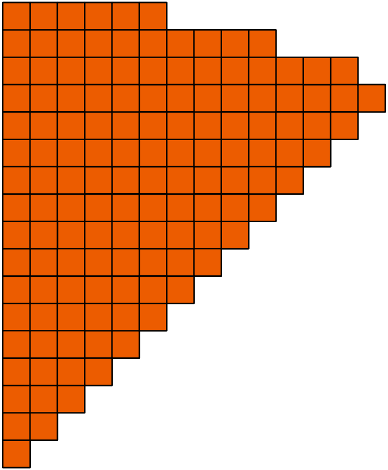

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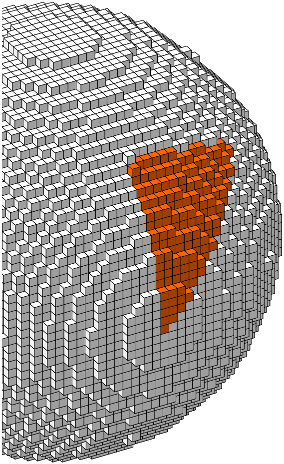
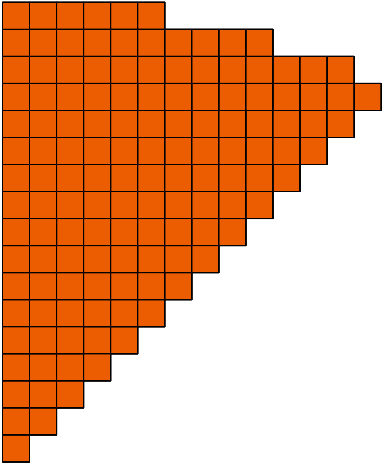




← 1-to-1 correspondence →
(on *xy*-plane)

Functional Plane

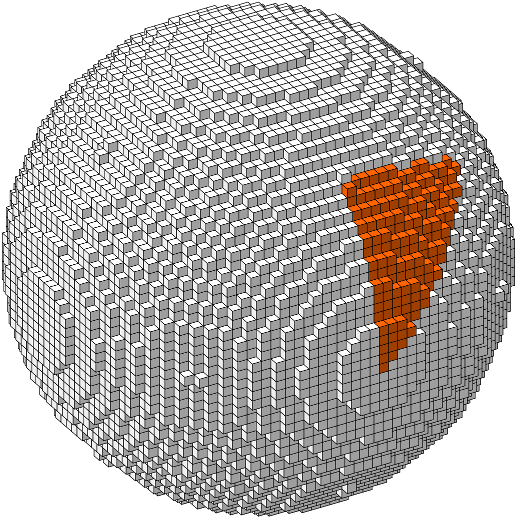
Definition (FP)

The functional plane of $\mathsf{S}_{r,t}$ is the coordinate plane on which its projection has a *bijection* with $\mathsf{S}_{r,t}$.

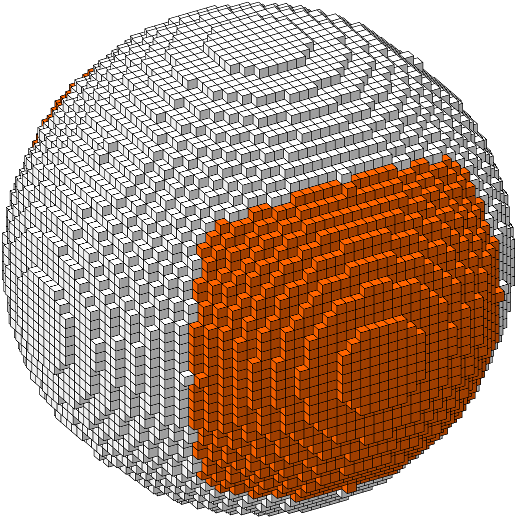
Lemma

FP of $\mathsf{S}_{r,t}$ *is xy-, yz-, or zx-plane, depending on whether the value of* $t \bmod 6$ *belongs to* $\{1, 2\}, \{3, 4\}$*, or* $\{5, 0\}$*, respectively.*

Functional Plane



Functional Plane



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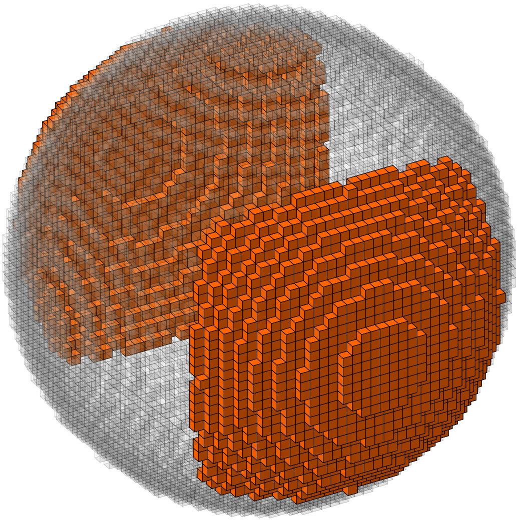
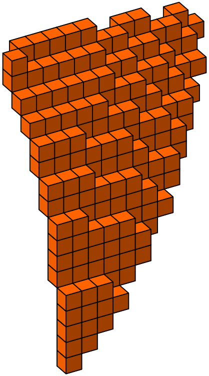
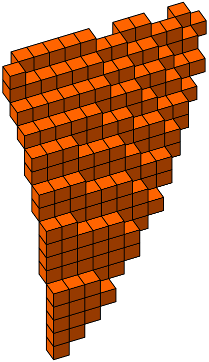


All Planes



All Planes



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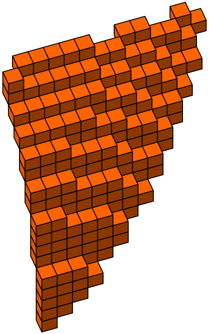
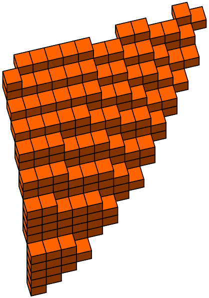
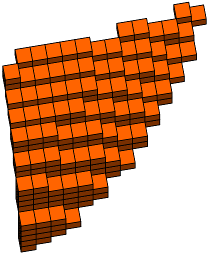


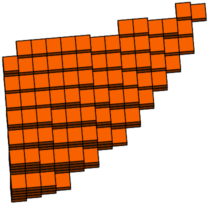
All Planes



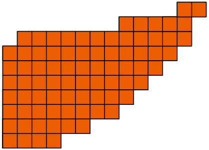
All Planes



All Planes

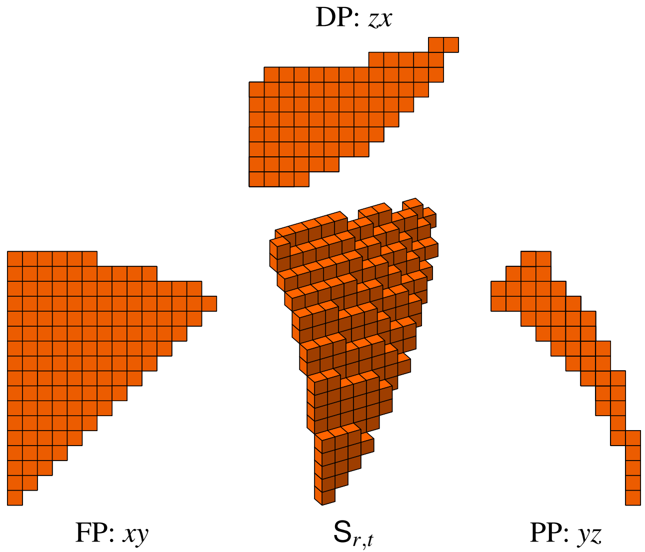


All Planes

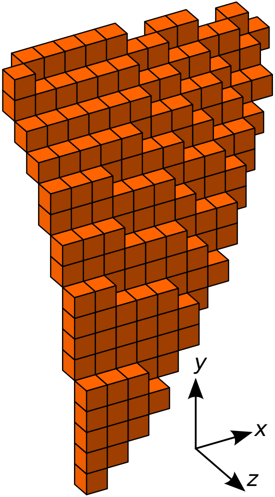


many-to-one correspondence
(on zx-plane)

All Planes



Jumps in Functional Plane



y \ x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
16	17	16	16	16	16	16								
15	17	17	17	17	17	17	16	16	15	15				
14	18	18	18	18	18	18	17	17	16	16	15	15	14	
13	19	19	19	19	19	18	18	18	17	17	16	15	15	14
12	20	20	20	19	19	19	19	18	18	17	17	16	16	
11	20	20	20	20	20	20	19	19	19	18	18	17		
10	21	21	21	20	20	20	20	19	19	19	18			
9	21	21	21	21	21	21	20	20	20	19				
8	22	22	21	21	21	21	21	20	20					
7	22	22	22	22	22	21	21	21						
6	22	22	22	22	22	22	21							
5	22	22	22	22	22	22								
4	23	23	23	22	22									
3	23	23	23	23										
2	23	23	23											
1	23	23												
0	23													

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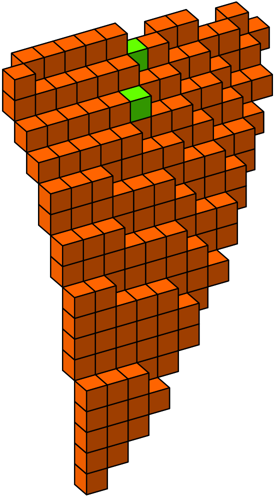

y \ x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
16	17	16	16	16	16	16								
15	17	17	17	17	17	17	16	16	15	15				
14	18	18	18	18	18	18	17	17	16	16	15	15	14	
13	19	19	19	19	19	18	18	18	17	17	16	15	15	14
12	20	20	20	19	19	19	19	18	18	17	17	16	16	
11	20	20	20	20	20	20	19	19	19	18	18	17		
10	21	21	21	20	20	20	20	19	19	19	18			
9	21	21	21	21	21	21	20	20	20	19				
8	22	22	21	21	21	21	21	20	20					
7	22	22	22	22	22	21	21	21						
6	22	22	22	22	22	22	21							
5	22	22	22	22	22	22								
4	23	23	23	22	22									
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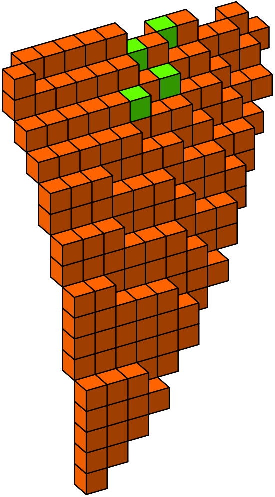
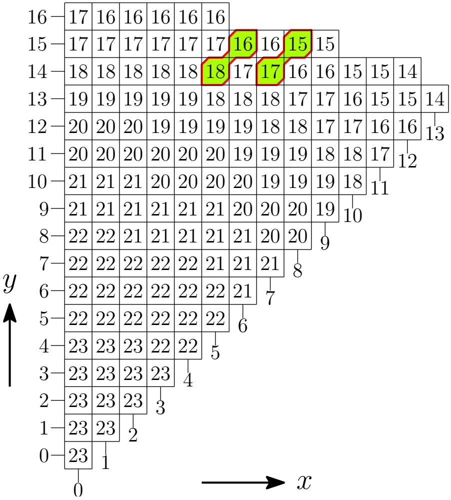
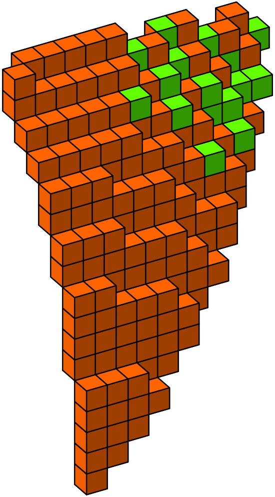
1	23	23												
0	23													

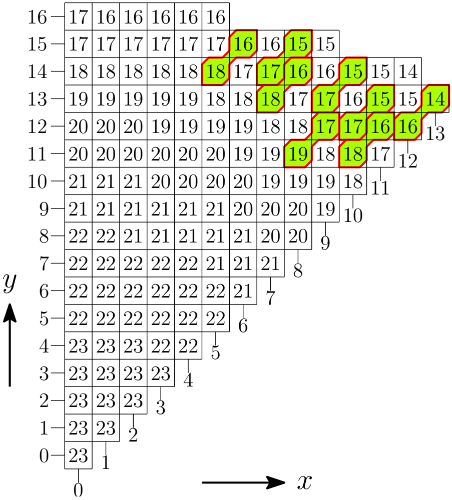
Jumps in Functional Plane



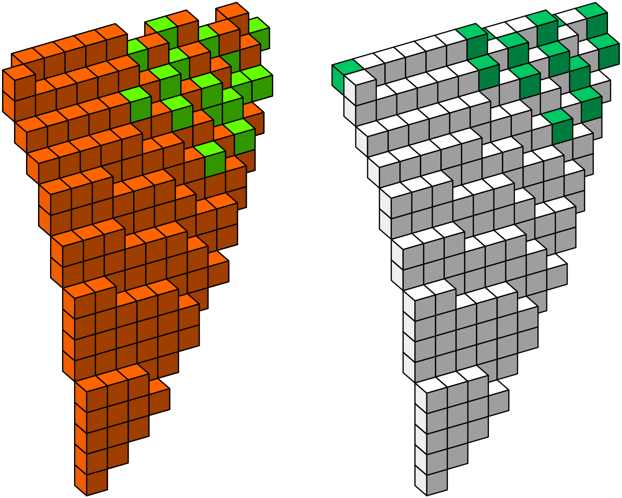


Jumps in Functional Plane

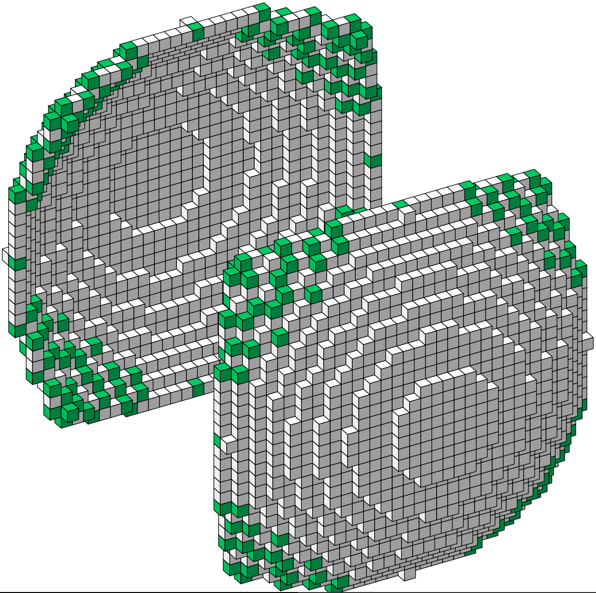




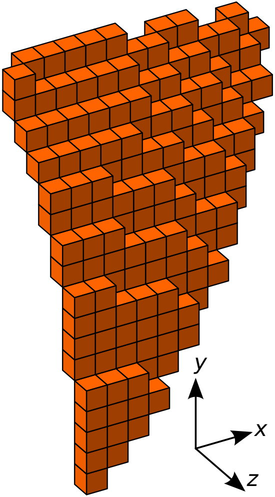
Jumps in Functional Plane

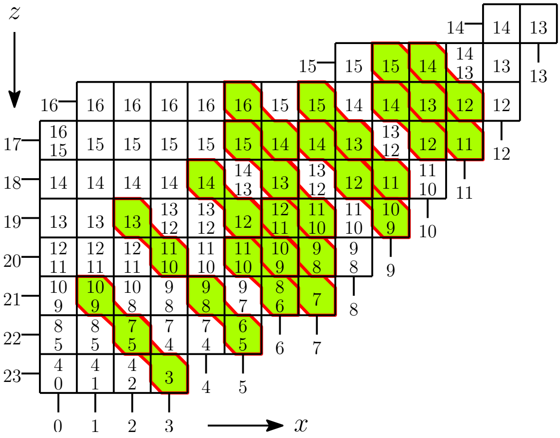


Jumps in Functional Plane

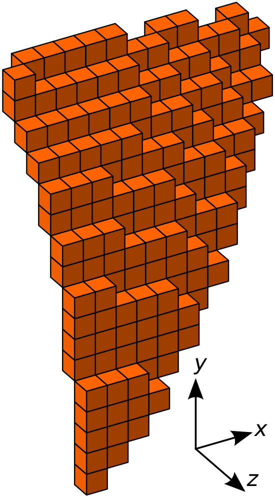


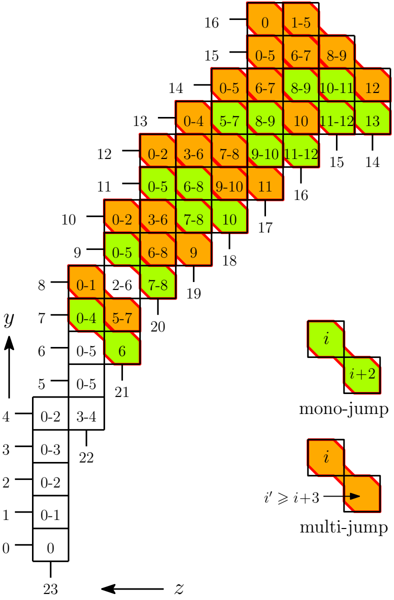
Jumps in Para-functional Plane





Jumps in Dia-functional Plane





FP, PP, DP

Definition (PP, DP)

PP of $\mathsf{S}_{r,t}$ is the coordinate plane on which its projection corresponds to only mono-jumps but is not bijective with $\mathsf{S}_{r,t}$.
Its DP is the coordinate plane which is neither FP nor PP.

Theorem (FP, PP Jumps)

Jumps corresponding to FP and PP are always mono-jumps; multi-jumps arise only in DP projection.

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Fixing FP, PP, DP

How to fix FP, PP, DP?

Example 1: $|x| \leqslant |y| \leqslant |z| \implies \text{FP} = xy, \text{PP} = zx, \text{DP} = yz.$

Example 2: $|x| \leqslant |z| \leqslant |y| \implies \text{FP} = zx, \text{PP} = xy, \text{DP} = yz.$

Example 3: $|y| \leqslant |z| \leqslant |x| \implies \text{FP} = yz, \text{PP} = xy, \text{DP} = zx.$

Lemma (Projection Planes)

FP, PP, and DP of $\mathsf{S}_{r,t}$ are determined by dropping from Q_t the coordinates $\omega[q_t^{(3)}]$, $\omega[q_t^{(2)}]$, and $\omega[q_t^{(1)}]$, respectively.

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Example 2: $|x| \leqslant |z| \leqslant |y| \implies \text{FP} = zx, \text{PP} = xy, \text{DP} = yz.$
Example 3: $|y| \leqslant |z| \leqslant |x| \implies \text{FP} = yz, \text{PP} = xy, \text{DP} = zx.$

Lemma (Projection Planes)

FP, PP, and DP of $\mathsf{S}_{r,t}$ are determined by dropping from Q_t the coordinates $\omega[q_t^{(3)}]$, $\omega[q_t^{(2)}]$, and $\omega[q_t^{(1)}]$, respectively.

Fixing FP, PP, DP

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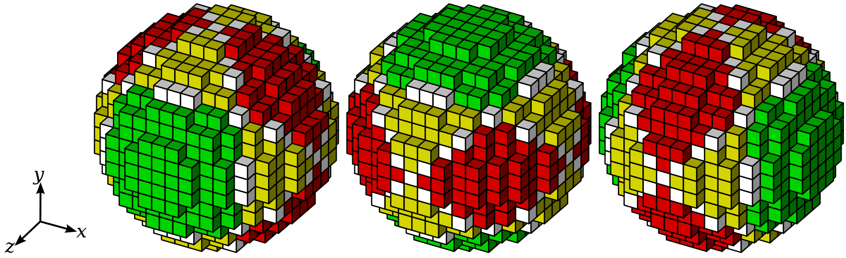
Q-octant Grouping

Proposition

For each $a \in N := \{1, 2, 3\}$, the group of q-octants with their common FP defined by $N \setminus \{a\}$ is $\mathcal{G}_a = \left\{ \mathsf{S}_{r,t} \middle| \omega[q_t^{(3)}] = a \right\}$; and their subgroups in $\mathcal{G}_a$ with common PP defined by $N \setminus \{b\}$ is $\mathcal{G}_{a:b} = \left\{ \mathsf{S}_{r,t} \middle| (\omega[q_t^{(3)}], \omega[q_t^{(2)}]) = (a, b) \right\}$, where $b \in N \setminus \{a\}$.



xy *zx* *yz*

FP-PP-DP = green-yellow-red

Application—Circle Drawing

Current status:

A limited research so far.

1. offset discretization scheme [1, 2]
2. discrete spherical paths and circles [3, 4, 5]

Application—Circle Drawing (2)

Definition

A (naive) *ortho-coordinate circle* $\mathsf{C}_r^{\langle a,b,c\rangle}$ is a discretization of the real circle $C_r^{\langle a,b,c\rangle}$ having radius r and lying on a real plane, with normal vector $\langle a, b, c\rangle$, that is orthogonal to one of the coordinate planes.

Considerations:

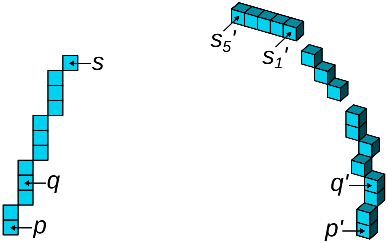
1. integer values for r, a, b, c
2. one of a, b, c is zero and the real plane passes through an integer point (w.l.o.g., $(0, 0, 0)$).

Application—Circle Drawing

Step 1

Let, w.l.o.g., $a = 0$. Set $p = (0,0)$, $q = (b,c)$ on yz-plane. Traverse digitally straight from p towards q up to s s.t. back projection of s on S_r has a voxel(s) with $x = 0$. [$b = 0 \implies y = 0, c = 0 \implies z = 0$.]

Example $(r = 12,\ a, b, c = 0, 1, -3)$



$x = 0$ (yz-plane) $\qquad \mathbb{Z}^3$

Application—Circle Drawing

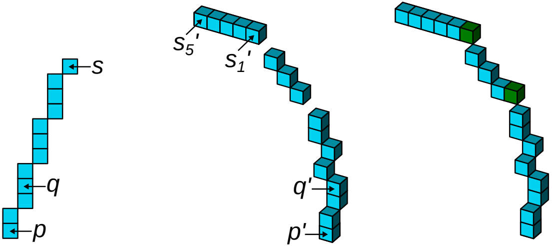
Step 2

For each pixel u in DSS(p, s), execute the following steps.

a) Use back projection from u to get a single voxel or a run of voxels on S_r.

b) If this voxel or voxel run is not connected with the last drawn voxel or voxel run, then identify the mono-jumps and the multi-jumps.

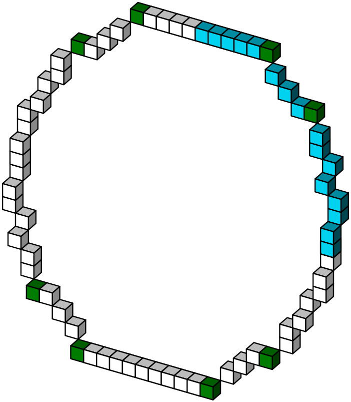
c) Compute and insert Steiner voxels for bridging the jumps. (Skip (b, c) if $u = p$.)



Application—Circle Drawing

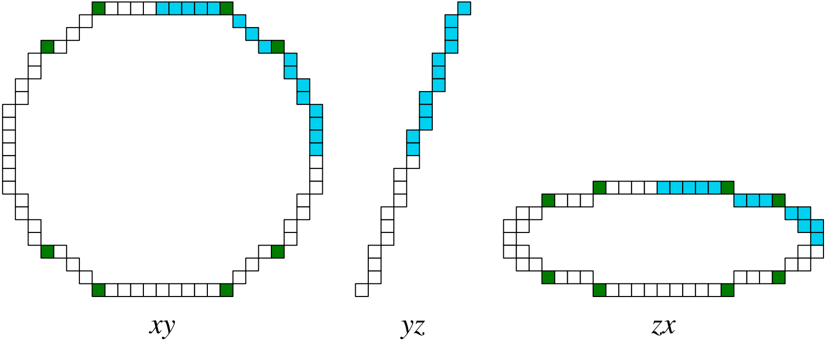
Step 3

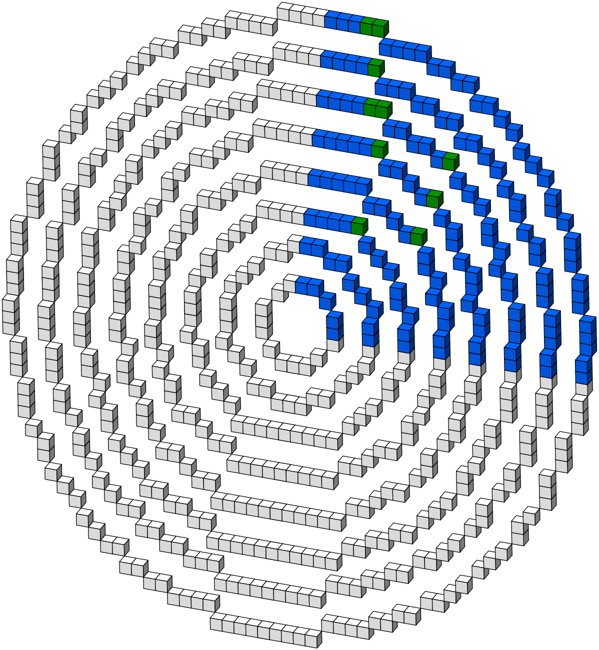
Use symmetry to construct the parts in other q-octants.



Application—Circle Drawing (6)

Projections





$a, b, c = 0, 1, -3;\ r = 3, 6, \ldots, 24.$

Challenges

1. Further generalization of 3D circle drawing.
2. Further simplification of sphere construction based on back projection.
3. Extension to higher dimensions.
4. Characterization of para-functional plane for naive planes and hyperplanes.

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