Construction of Digital Ellipse by Recursive Integer Intervals

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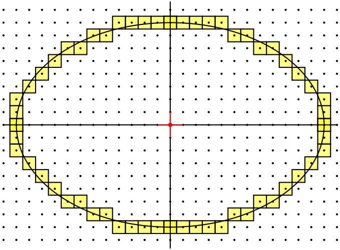
Digital Ellipse

Definition

$\mathsf{E}(a,b)$ = 0-connected irreducible sequence of integer points within isothetic distance $\frac{1}{2}$ from the real ellipse.[1]

[1] a and b are integer lengths of semi-major and semi-minor axes, which are axis-parallel, and center is $(0,0)$.



$a = 12, b = 8$

Examples

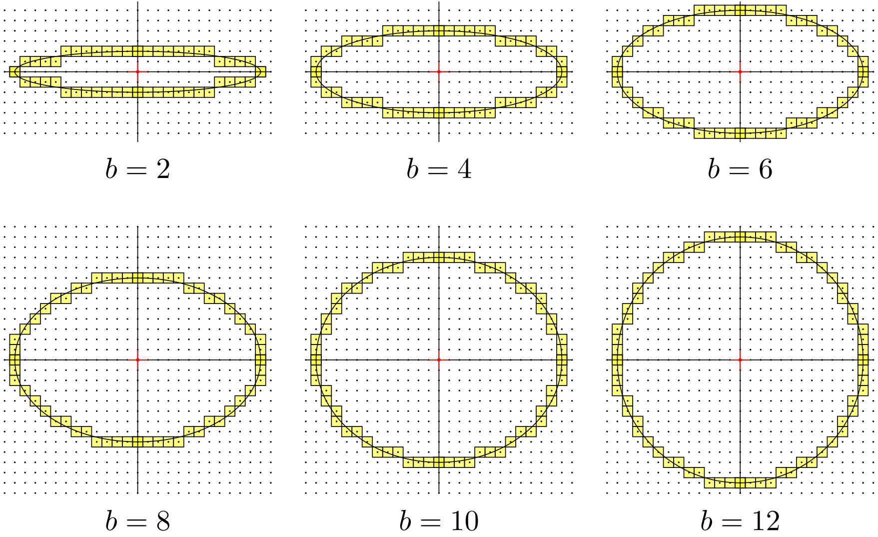
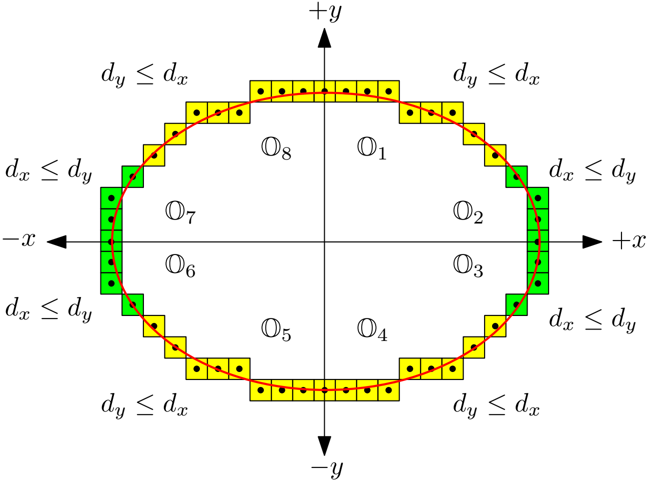


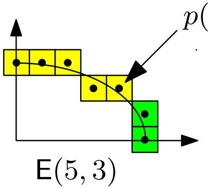
Figure: Digital ellipses with $a = 12$ and increasing values of b.

Elliptic Octants: $\mathbb{O}_1, \mathbb{O}_2, \ldots, \mathbb{O}_8$



Theorem (First e-octant points)

An integer point $p(i,j)$ *with* $j > 0$ *belongs to* $\mathsf{E}_1(a,b)$ *if and only if* $4b^2a^2 - a^2(2j+1)^2 \leqslant 4b^2i^2 \leqslant 4b^2a^2 - a^2(2j-1)^2 - 1$.



$p(4,2) \implies i = 4, j = 2.$

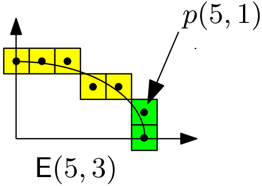
$4b^2a^2 - a^2(2j+1)^2 = 275.$

$4b^2a^2 - a^2(2j-1)^2 - 1 = 674.$

$4b^2i^2 = 576 \in [275, 674].$

Corollary (Second e-octant points)

An integer point (i, j) *with* $i > 0$ *belongs to* $\mathsf{E}_2(a, b)$ *if and only if* $4b^2a^2 - b^2(2i+1)^2 \leqslant 4a^2j^2 \leqslant 4b^2a^2 - b^2(2i-1)^2 - 1$. *Moreover, if* $p(i, j)$ *belongs to* $\mathsf{E}_2(a, b)$ *but does not satisfy the equation, then* $i = 0$.



$$p(5,1) \implies i = 5, j = 1.$$
$$4b^2a^2 - b^2(2i+1)^2 = -189.$$
$$4b^2a^2 - b^2(2i-1)^2 - 1 = 170.$$
$$4a^2j^2 = 100 \in [-189, 170].$$

Interval Recurrences

Theorem (First e-Octant)

An integer point (i, j) *with* $j > 0$ *belongs to* $\mathsf{E}_1(a, b)$ *if and only if* $4b^2i^2$ *lies in the interval* $I_k = [u_k, v_k := u_k + l_k - 1]$*, where* $j = b - k$*,* $k \geqslant 0$*, and* u_k *and* l_k *are given as follows.*

$$u_k = \begin{cases} -4a^2b - a^2 & \text{if } k = 0 \\ u_{k-1} + l_{k-1} & \text{otherwise} \end{cases}$$

$$l_k = \begin{cases} 8a^2b & \text{if } k = 0 \\ l_{k-1} - 8a^2 & \text{otherwise} \end{cases}$$

Corollary (Second e-Octant)

An integer point (i, j) *with* $i > 0$ *belongs to* $\mathsf{E}_2(a, b)$ *if and only if* $4a^2j^2$ *lies in the interval* $I_k = [u_k, v_k := u_k + l_k - 1]$, *where* $i = a - k$, $k \geqslant 0$, *and* u_k *and* l_k *are given as follows.*

$$u_k = \begin{cases} -4b^2a - b^2 & \text{if } k = 0 \\ u_{k-1} + l_{k-1} & \text{otherwise} \end{cases}$$

$$l_k = \begin{cases} 8b^2a & \text{if } k = 0 \\ l_{k-1} - 8b^2 & \text{otherwise} \end{cases}$$

Algorithm: DRAW-ELLIPSE-INT

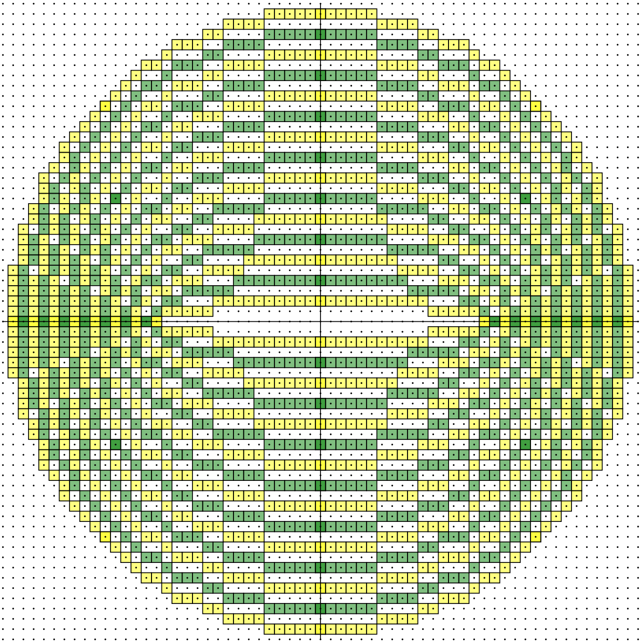
Input: int a, b
Output: Digital ellipse

int $i \leftarrow 0, j \leftarrow b, u \leftarrow -4a^2b - a^2, l \leftarrow 8a^2b, v \leftarrow 4a^2b - a^2 - 1, s \leftarrow 0, i_1, j_1$ ▷ `initialization for e-octant 1`
$\mathsf{E} \leftarrow \{\}$(▷ `output set of integer points`)
while $(u \leqslant s) \wedge (s \leqslant v)$ **do** ▷ `runs genarator for e-octant 1`
    `drawRunInt`$(i, j, a, b, 1, s, u, l, v, i_1, j_1, \mathsf{E})$ ▷ `points for e-octant1`
$i_1 \leftarrow i - 1, j_1 \leftarrow j + 1$
while $(u > v) \wedge (i < a)$ **do** ▷ `for e-octant 1 at j=0`
    `include4SymPoints`(i, j, E) $i_1 \leftarrow i, j_1 \leftarrow j, i \leftarrow i + 1, s \leftarrow 4b^2i^2$
$i \leftarrow a, j \leftarrow 0, u \leftarrow -4b^2a - b^2, l \leftarrow 8b^2a, v \leftarrow 4b^2a - b^2 - 1, s \leftarrow 0$
▷ `initialization for e-octant 1`
while $(u \leqslant s) \wedge (s \leqslant v) \wedge ((i \neq i_1) \vee (j \neq j_1))$ **do** ▷ `runs genarator for e-octant 2`
    `drawRunInt`$(j, i, b, a, 2, s, u, l, v, i_1, j_1, \mathsf{E})$ ▷ `points for e-octant1`
while $(u > v) \wedge (j < b)$ **do** ▷ `for e-octant 1 at i=0`
    `include4SymPoints`(i, j, E) $j \leftarrow j + 1, s \leftarrow 4a^2j^2$
return E

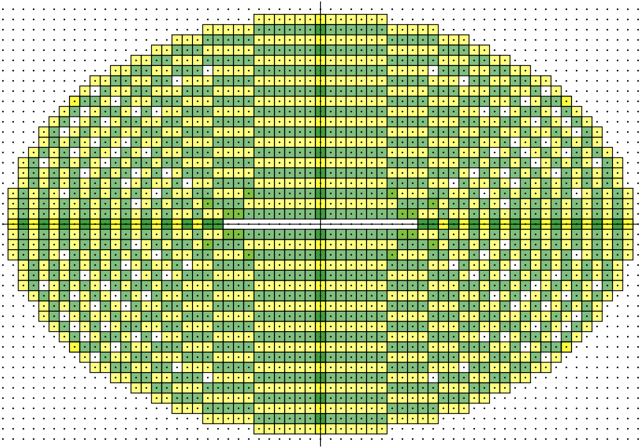
Procedure

Procedure `drawRunInt`$(i, j, a, b, t, s, u, l, v, i_1, j_1, \mathsf{E})$

repeat ▷ points generator for k-th run
  if $t = 1$ **then** `include4SymPoints`(i, j, E) ▷ include points for e-octant 1,4,5,8
  else if $(i = i_1) \wedge (j = j_1)$ **then return**
    else `include4SymPoints`(j, i, E) ▷ include points for e-octant 2,3,6,7
  $i \leftarrow i + 1, s \leftarrow 4b^2 i^2$ ▷ updation for next point
until $s > v$
$j \leftarrow j - 1$
$u \leftarrow v + 1, l \leftarrow l - 8a^2, v \leftarrow u + l - 1$ ▷ updation for k+1-th run



$$(a, b) = (16, 2), (17, 4), (18, 6), \ldots, (30, 30)$$



$$(a, b) = (11, 1), (12, 2), (13, 3), \ldots, (30, 20).$$

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