# From circle to sphere and to related problems in the digital space

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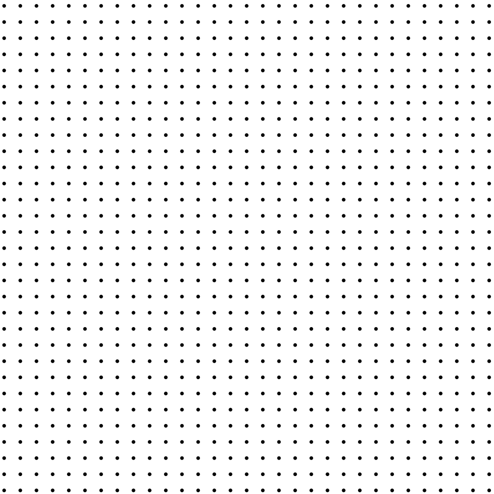
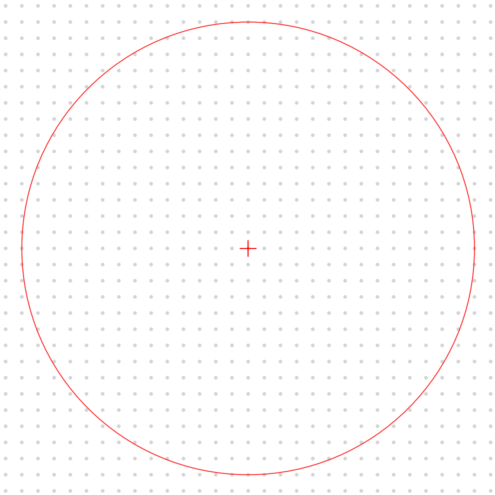
# Circle Construction

# Construction by Digitization



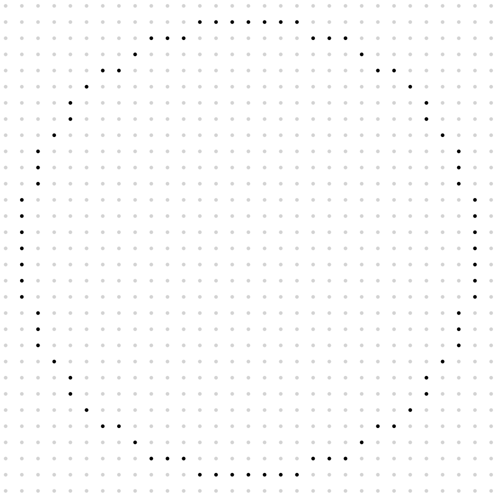
This is $\mathbb{Z}^2$ — an infinite set of 2D integer points

# Construction by Digitization



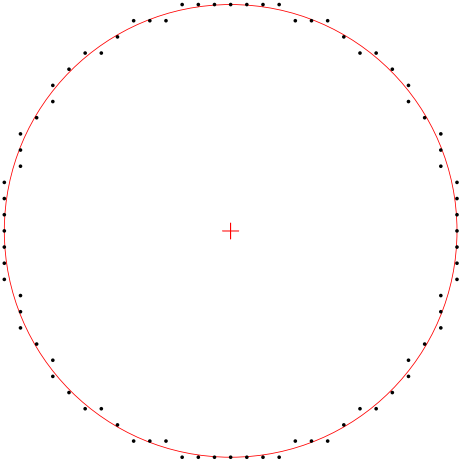
This is a real circle
(integer center, radius 14)

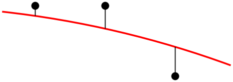
# Construction by Digitization



This is the *digital circle* (integer center, radius 14)

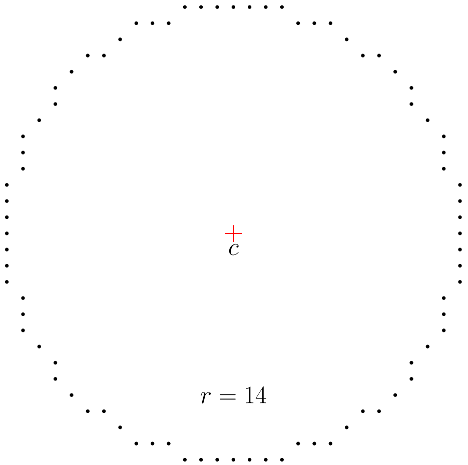
# Construction by Digitization





$$\text{distance} < \frac{1}{2}$$

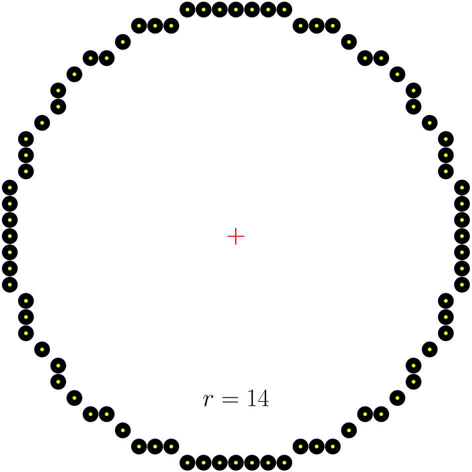
# Construction by Digitization




**The challenge**

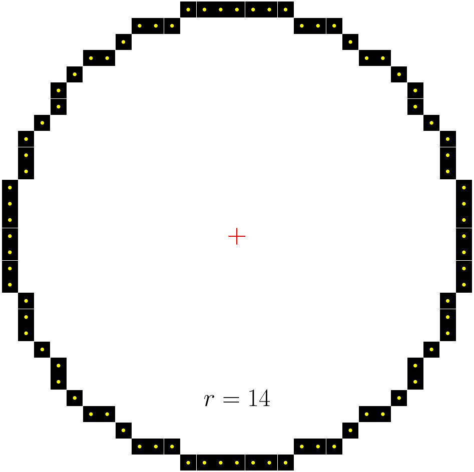
Given $r$ and $c$ as integers, use only *integer arithmetic* to compute the digital circle. (w.l.o.g., $c = (0, 0)$)

# Construction by Digitization




As displayed on computer screen / book

# Construction by Digitization



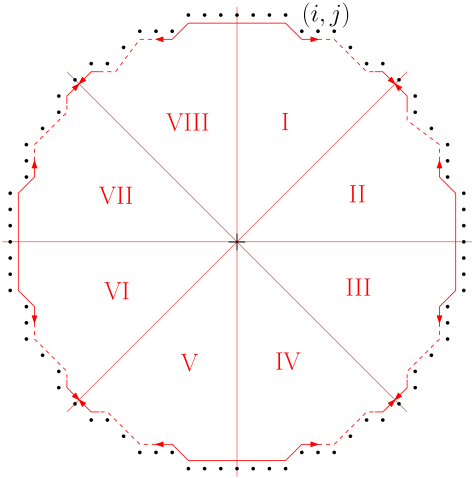

As displayed on computer screen / book

# Construction by Digitization

| Algorithm | Inventors | Year |
|---|---|---|
| Incremental | Bresenham | 1977 |
| Optimized midpoint | Foley *et al.* | 1993 |
| Short run | Hsu *et al.* | 1993 |
| Hybrid run slice | Yao & Rokne | 1995 |
| *Number theory* [a] | Bhowmick & Bhattacharya | 2008 |

[a] P. Bhowmick and B. B. Bhattacharya, Number-theoretic interpretation and construction of a digital circle, *Discrete Applied Mathematics*, **156** : 2381–2399, **2008.**

# Octant Property




#permutations of $(i, j)$ including sign $= 2 \times 2^2 = 8$.

# Number-theoretic Properties

A simple question: What's the pattern here? (Disregard the 1st line)

```
 0, 13
14, 39
40, 63
64, 85
86,105
  ...
```

# Number-theoretic Properties

A simple question: What's the pattern here? (Disregard the 1st line)

| | | length |
|---|---|---|
| 0, 13 | | |
| 14, 39 | → | 26 |
| 40, 63 | → | 24 |
| 64, 85 | → | 22 |
| 86,105 | → | 20 |
| ... | | |

# Number-theoretic Properties

A simple question: What's the pattern here? (Disregard the 1st line)

```
 0, 13
14, 39  →  26
                } → −2
40, 63  →  24
                } → −2
64, 85  →  22
                } → −2
86,105  →  20
   ...
```

# Number-theoretic Properties

A simple question: What's the pattern here? (Disregard the 1st line)

```
 0, 13
14, 39  →  26 }
40, 63  →  24 }  →  −2
               }  →  −2
64, 85  →  22 }
86,105  →  20 }  →  −2
   ...
```

# Number-theoretic Properties

A simple question: What's the pattern here? (Disregard the 1st line)

| | | | |
|---|---|---|---|
| 0, 13 | | | |
| 14, 39 | → 26 | | |
| | | → $-2$ | |
| 40, 63 | → 24 | | |
| | | → $-2$ | This is how we get the digital circle for $r = 14$ |
| 64, 85 | → 22 | | |
| | | → $-2$ | |
| 86,105 | → 20 | | |
| ... | | | |

# Number-theoretic Properties

A simple question: What's the pattern here? (Disregard the 1st line)

```
0, 13
14, 39   →  26
40, 63   →  24   } → −2
64, 85   →  22   } → −2
86,105   →  20   } → −2
...
```

$I_0 = [0, r-1]$

$I_1 = [r, 3r-3]$

$I_2 = [3r-2, 5r-7]$

$\ldots$

$I_k = [u_k, u_k + l_k - 1]$

$\ldots$

# Number-theoretic Properties

A simple question: What's the pattern here? (Disregard the 1st line)

| | | |
|---|---|---|
| 0, 13 | | |
| 14, 39 | → 26 | |
| | | → −2 |
| 40, 63 | → 24 | |
| | | → −2 |
| 64, 85 | → 22 | |
| | | → −2 |
| 86,105 | → 20 | |
| ... | | |

$$I_0 = [0, r-1]$$

$$I_1 = [r, 3r-3]$$

$$I_2 = [3r-2, 5r-7]$$

$$\ldots$$

$$I_k = [u_k, u_k + l_k - 1]$$

$$\ldots$$

Recurrence for $r$

$$u_k = \begin{cases} u_{k-1} + l_{k-1} & \text{if } k \geq 1 \\ 0 & \text{if } k = 0 \end{cases}$$

$$l_k = \begin{cases} l_{k-1} - 2 & \text{if } k \geq 2 \\ 2r - 2 & \text{if } k = 1 \\ r & \text{if } k = 0 \end{cases}$$

# Number-theoretic Properties

A simple question: What's the pattern here? (Disregard the 1st line)

```
 0,  13
14,  39  →  26
40,  63  →  24
64,  85  →  22
86,105   →  20
  ...
```

$26 \to 24$: $-2$; $24 \to 22$: $-2$; $22 \to 20$: $-2$

$I_0 = [0, r-1]$

$I_1 = [r, 3r-3]$

$I_2 = [3r-2, 5r-7]$

$\dots$

$I_k = [u_k, u_k + l_k - 1]$

$\dots$

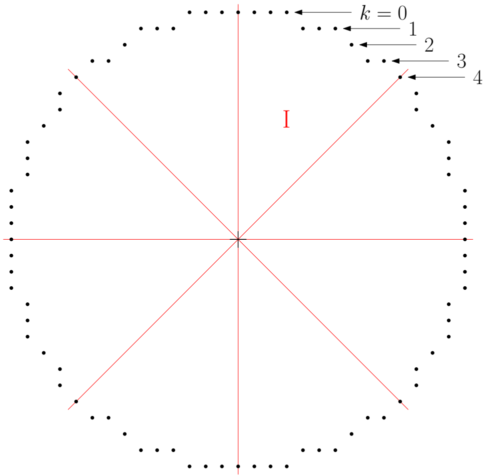
Recurrence for $r$

$$u_k = \begin{cases} u_{k-1} + l_{k-1} & \text{if } k \geq 1 \\ 0 & \text{if } k = 0 \end{cases}$$

$$l_k = \begin{cases} l_{k-1} - 2 & \text{if } k \geq 2 \\ 2r-2 & \text{if } k = 1 \\ r & \text{if } k = 0 \end{cases}$$

**Theorem** $I_k$ contains the $x^2$ of the points at $k$th run.

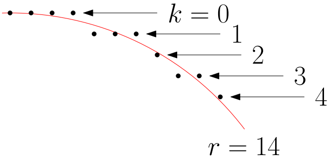
# Number-theoretic Properties (1)




$r = 14$ :
$k = 0 : [0, 13] \Rightarrow 4$ squares
$k = 1 : [14, 39] \Rightarrow 3$ squares
$k = 2 : [40, 63] \Rightarrow 1$ square
$k = 3 : [64, 85] \Rightarrow 2$ squares
$k = 4 : [86, 105] \Rightarrow 1$ square

# Algorithm **DCS** (`int` $r$)

```
int i ← 0, j ← r, s ← 0, w ← r − 1, l ← 2r − 2
while j ⩾ i
    do
        select (i, j)
        s ← s + 2i + 1
        i ← i + 1
    while s ⩽ w
    w ← w + l
    l ← l − 2
    j ← j − 1
```




$$k = 0 : [0, r-1] = [0, 13] \Rightarrow 4$$
$$k = 1 : [r, 3r-3] = [14, 39] \Rightarrow 3$$
$$k = 2 : [3r-2, 5r-7] = [40, 63] \Rightarrow 1$$
$$k = 3 : [5r-6, 7r-13] = [64, 85] \Rightarrow 2$$
$$k = 4 : [7r-12, 9r-21] = [86, 105] \Rightarrow 1$$

# More number-theoretic properties (1)

## A simple observation


Let $u, v, w$ be three positive integers in increasing order such that $w - v = v - 1 - u$.

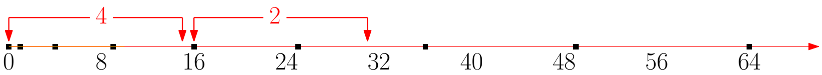
Let

$s[u, v-1] =$ #squares in $[u, v-1]$,

$s[v, w] =$ #squares in $[v, w]$.

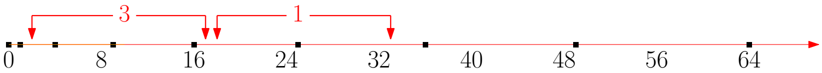
Then can $s[v, w] > s[u, v-1]$? If so, by how much?

# More number-theoretic properties (2)

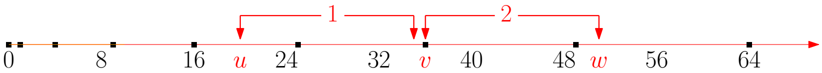


# More number-theoretic properties





# More number-theoretic properties (4)




**Lemma**

*For $u < v < w$ and $w - v = v - 1 - u$, $s[v, w] \leqslant s[u, v-1] + 1$.*

**Hence, a useful result:**

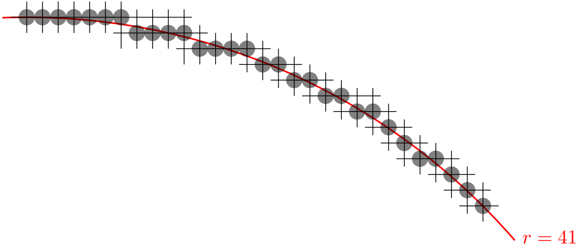
For $u < v < w$ and $w - v = v - u - 3$, $s[v, w] \leqslant s[u, v-1] + 1$.

And so the theorem follows in next slide!

# More number-theoretic properties (5)

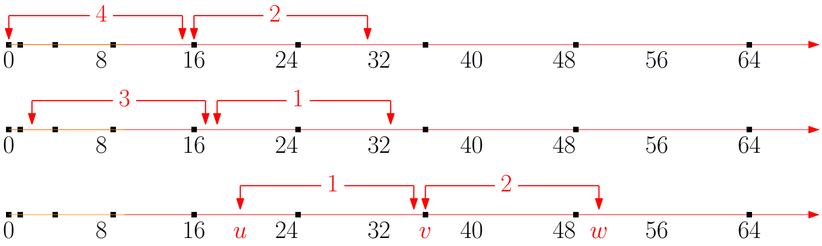
## Theorem (Upper bound of run length ($\lambda$))

$$\lambda(j-1) \leqslant \lambda(j) + 1.$$



# More number-theoretic properties






**Lemma**

*For* $u < v < w$ *and* $w - v = v - 1 - u$, $s[v, w] \geqslant \left\lfloor \frac{s[u,v-1]-1}{2} \right\rfloor$.

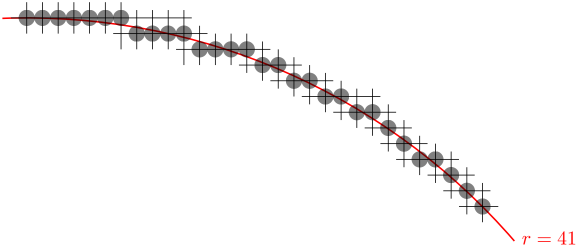
And so the theorem follows in next slide!

# More number-theoretic properties



## Theorem (Lower bound of run length ($\lambda$))

$$\lambda(j-1) \geqslant \left\lfloor \frac{\lambda(j)-1}{2} \right\rfloor - 1$$

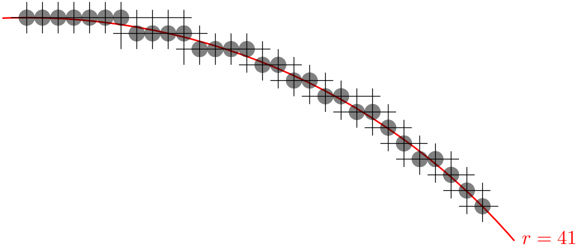


# More number-theoretic properties

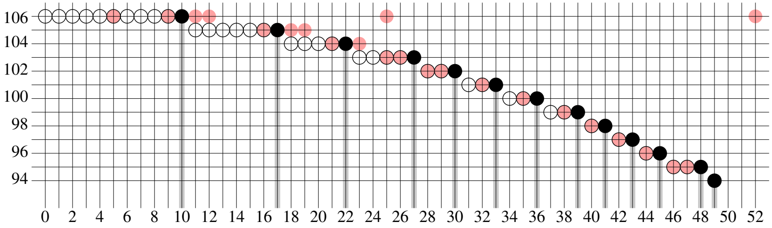


## Constructive bounds

$$\left\lfloor \frac{\lambda(j)-1}{2} \right\rfloor - 1 \leqslant \lambda(j-1) \leqslant \lambda(j) + 1$$



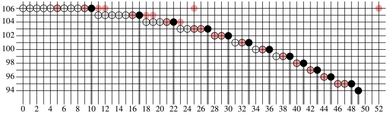
# Algorithm **DCR**




Demonstration of **DCR** for $r = 106$.

# Algorithm **DCR**: Square search

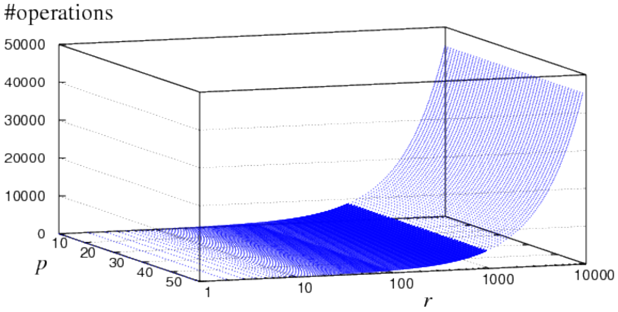
```
Algorithm DCR (int r) {
  int i = 0, j = r, w = r − 1, m;
  int s = 0, t = r, l = w << 1;
  while (j ⩾ i) {
        while (s < t) {
              m = s + t;
              m = m >> 1;
              if (w ⩽ square[m])
                    t = m;
              else
                    s = m + 1; }
        if (w < square[s])
              s − −;
        s + +;
        include_run (i, s − i, j);
        t = s + s − i + 1;
        i = s;
        w = w + l;
        l = l − 2;
        j − −; }}
```



# Hybrid algorithm **DCH** (1)

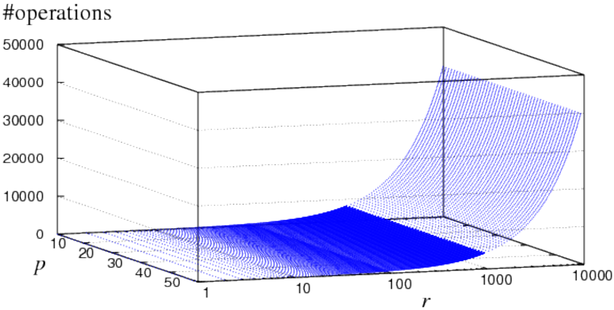
```
Algorithm DCH (int r, int p) {
  int i = 0, j = r, w = r − 1, m;
  int s = 0, t = r, l = w << 1;
  while (j ⩾ i) {
     while (s < t) {
        m = s + t;
        m = m >> 1;
        if (w ⩽ square[m])
           t = m;
        else
           s = m + 1; }
     if (w < square[s])
        s − −;
     s + +;
     include_run (i, s − i, j);
     if (s − i < p)
        break;
     t = s + s − i + 1;
     i = s;
     w = w + l;
     l = l − 2;
     j − −; }
  i = s − 1;
  s = square[s];
  w = w + l;
  l = l − 2;
  j − −;
  while (j ⩾ i) {
     do {sym_8 (i, j);
        s = s + i;
        i + +;
        s = s + i; } while (s ⩽ w);
     w = w + l;
     l = l − 2;
     j − −; }}
```

# Test Results...



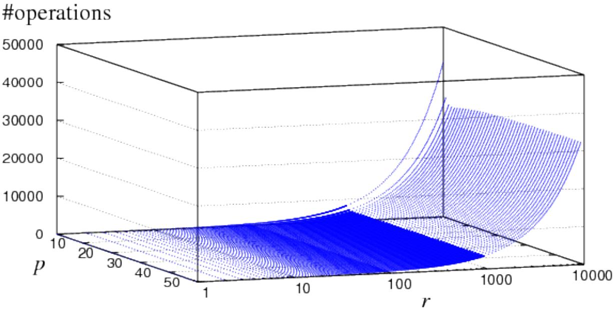

DCB

# Test Results...




DCR

# Test Results...

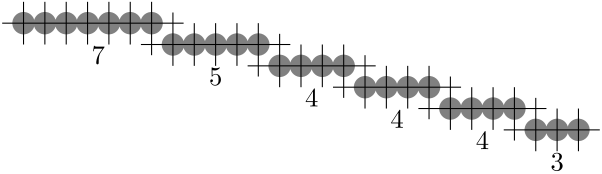



DCH

# Digital Circularity

# Problem Statement

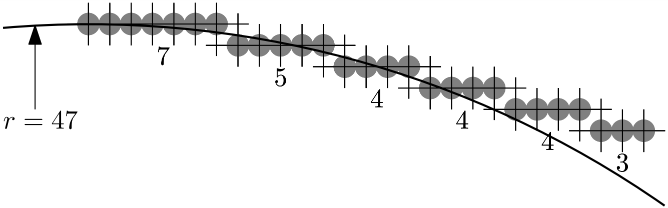





Does there exist a real circle (integer radius & center) such that each point of the given sequence lies within a distance of $\frac{1}{2}$ from it?

# Problem Statement

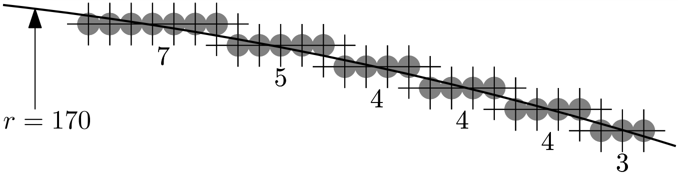





47 is far from true.
Seems, it will be much larger!
But how large? And how to get it?

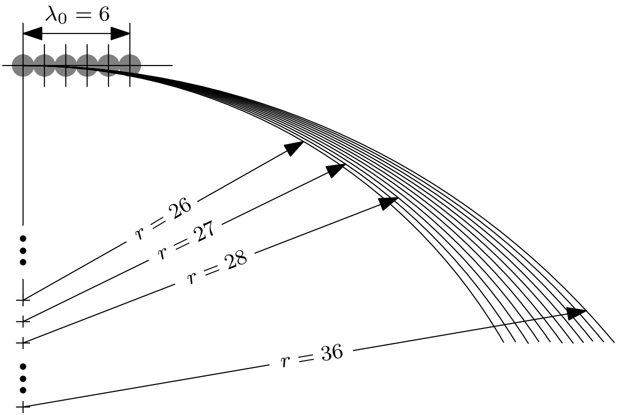
# Problem Statement






170 is *the* solution!
How to get it very fast, using simple arithmetic (no trigonometry etc.)?

# Conflicting Radii (1)




$r \in [26, 36]$

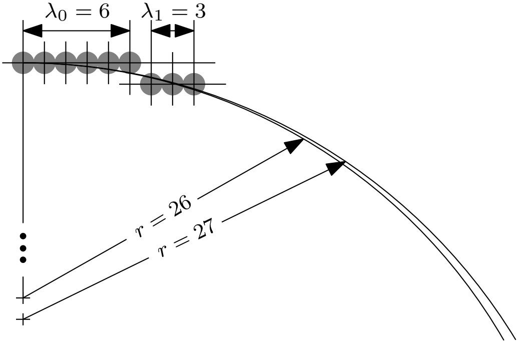
# Conflicting Radii (2)

**Lemma**

*$\lambda_0$ is the length of top run of a digital circle $\mathcal{C}^{\mathbb{Z}}(o, r)$ iff $r \in R_0 := [(\lambda_0 - 1)^2 + 1, \lambda_0^2]$.*

# Conflicting Radii

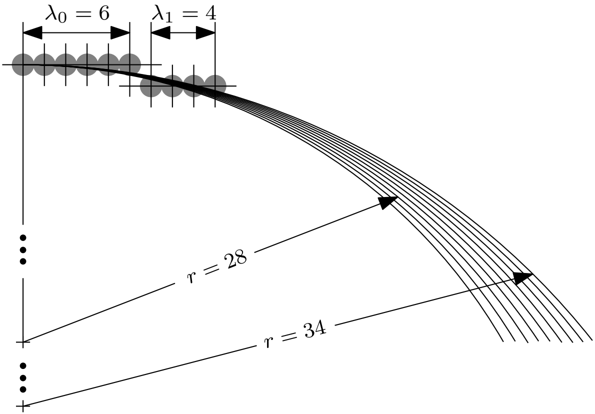





$r \in [26, 27]$

# Conflicting Radii

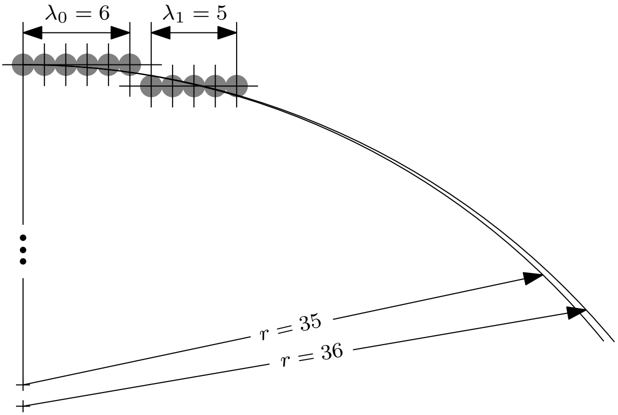





$r \in [28, 34]$

# Conflicting Radii

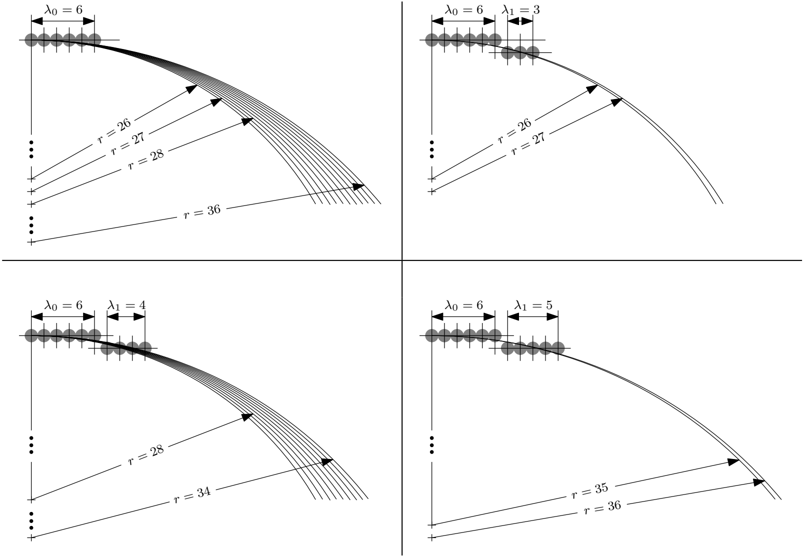





$r \in [35, 36]$

# Radii Nesting





# Radii Nesting (2)

## Lemma

*$\lambda_0$ and $\lambda_1$ are the lengths of top two runs of $\mathcal{C}^{\mathbb{Z}}(o,r)$ iff $r \in R_0 \cap R_1$, where, $R_1 = \left[ \left\lceil \frac{(\Lambda_1 - 1)^2 + 3}{3} \right\rceil, \left\lfloor \frac{\Lambda_1^2 + 2}{3} \right\rfloor \right]$, $\Lambda_1 = \lambda_0 + \lambda_1$.*

*(If $R_0 \cap R_1 = \emptyset$, then there exists no digital circle ... $\lambda_0$ and $\lambda_1$.)*

# Radii Nesting



## Theorem (Radii interval)

$\langle \lambda_0, \ldots, \lambda_n \rangle$ *is the sequence of top* $n+1$ *run-lengths of* $\mathcal{C}^{\mathbb{Z}}(o,r)$ *iff*

$$r \in \bigcap_{k=0}^{n} R_k$$

*where,*

$$R_k = \left[ \left\lceil \frac{1}{2k+1} \left( (\Lambda_k - 1)^2 + k(k+1) + 1 \right) \right\rceil, \left\lfloor \frac{1}{2k+1} \left( \Lambda_k^2 + k(k+1) \right) \right\rfloor \right]$$

*and*

$$\Lambda_k = \sum_{j=0}^{k} \lambda_j .$$

*(If* $\bigcap_{k=0}^{n} R_k = \emptyset$*, then there exists no digital circle whose top* $n+1$ *runs have length* $\langle \lambda_0, \lambda_1, \ldots, \lambda_n \rangle$*.)*

# Algorithm DCT

$\Lambda \leftarrow S[0]$

$[r', r''] \leftarrow [(\Lambda - 1)^2 + 1, \Lambda^2]$

**for** $k \leftarrow 1$ to $n - 1$

    $\Lambda \leftarrow \Lambda + S[k]$

    $s' \leftarrow \lceil ((\Lambda - 1)^2 + k(k+1) + 1)/(2k+1) \rceil$

    $s'' \leftarrow \lfloor (\Lambda^2 + k(k+1))/(2k+1) \rfloor$

    **if** $s'' < r'$ **or** $s' > r''$

        **print** "$S$ is circular up to $(k-1)$th run for $[r', r'']$."

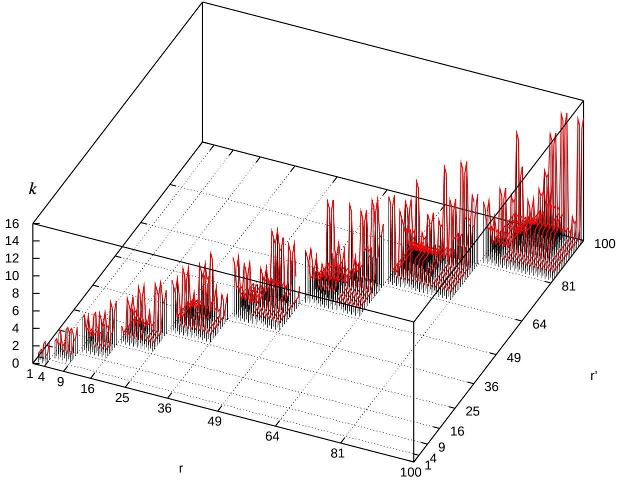
        **return**

    **else**

        $[r', r''] \leftarrow [\max(r', s'), \min(r'', s'')]$

**print** "$S$ is circular in entirety for $[r', r'']$."

# Conflicting Radii: Resolved how fast? (1)

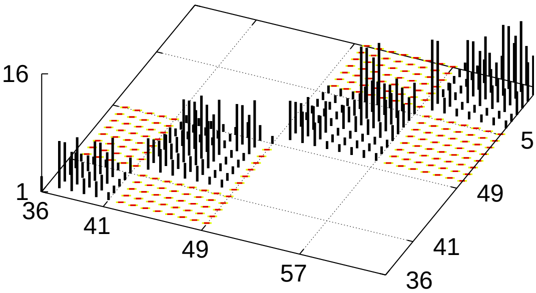


# Conflicting Radii: Resolved how fast? (2)

Conflicting radii starting from $k = 0$

# Conflicting Radii: Resolved how fast? (3)

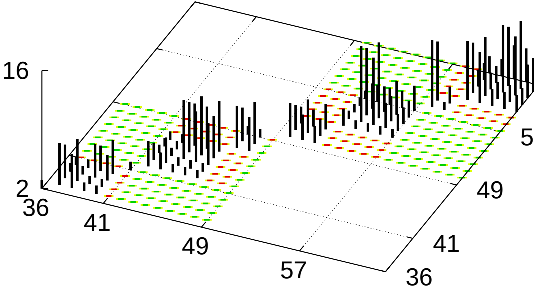
Resolving the conflicting radii $r'$ with increasing $k$




$k = 1$

# Conflicting Radii: Resolved how fast? (4)

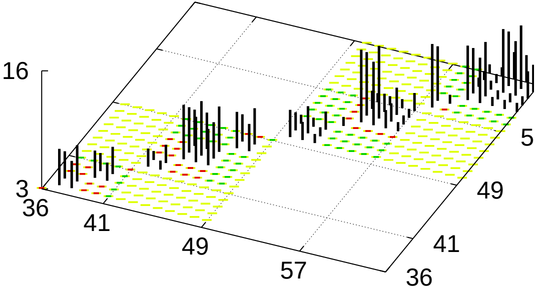
Resolving the conflicting radii $r'$ with increasing $k$




$k = 2$

# Conflicting Radii: Resolved how fast? (5)

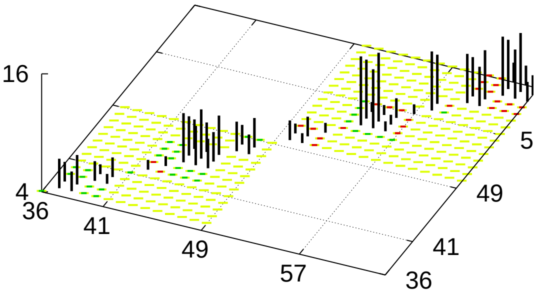
Resolving the conflicting radii $r'$ with increasing $k$




$k = 3$

# Conflicting Radii: Resolved how fast? (6)

Resolving the conflicting radii $r'$ with increasing $k$




$k = 4$

# General Case & DCG



## Lemma

*If a digital circle of radius r contains a given run of length $\lambda$, then there exist two positive integers a and k such that* $r \geqslant \lceil \max\left(f_{1,\lambda}(a,k), f_{2,\lambda}(a,k)\right) \rceil$, *where*

$$f_{1,\lambda}(a,k) = \frac{(a-1)^2 + k(k-1) + 1}{2k-1}$$

*and*

$$f_{2,\lambda}(a,k) = \frac{(a+\lambda-1)^2 + k(k+1) + 1}{2k+1}.$$

# General Case & DCG



**Lemma**

*If a digital circle of radius r contains a given run of length $\lambda$, then there exist two positive integers a and k such that $r \leqslant \lfloor \min\left(f_{3,\lambda}(a,k), f_{4,\lambda}(a,k)\right) \rfloor$, where*

$$f_{3,\lambda}(a,k) = \frac{a^2 + k(k-1)}{2k-1}$$

*and*

$$f_{4,\lambda}(a,k) = \frac{(a+\lambda)^2 + k(k+1)}{2k+1}.$$

# General Case & DCG



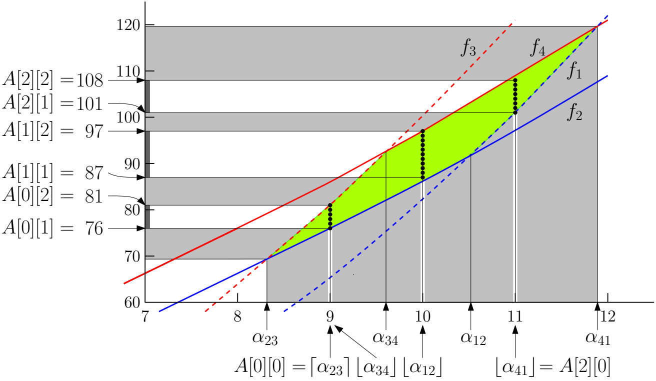
## Theorem

*An arbitrary run of given length $\lambda$ belongs to only those digital circles whose radii are in the range*

$$\mathcal{R}_{ak} = \begin{array}{c} \left\{ r \mid r \geqslant \left\lceil \max\limits_{a,k \in \mathbb{Z}^+} \left( f_{1,\lambda}(a,k), f_{2,\lambda}(a,k) \right) \right\rceil \right\} \\ \bigcap \\ \left\{ r \mid r \leqslant \left\lfloor \min\limits_{a,k \in \mathbb{Z}^+} \left( f_{3,\lambda}(a,k), f_{4,\lambda}(a,k) \right) \right\rfloor \right\}. \end{array}$$

# General Case & DCG





# General Case & DCG



Points of intersection (in $\mathbb{R}^2$) among the parabolas $\{f_{i,\lambda} \mid i = 1, 2, 3, 4\}$ defining $\mathcal{R}_{ak}$.

$$(\underline{k} = 2k - 1, \overline{k} = 2k + 1, \hat{\underline{k}} = k(k-1), \hat{\overline{k}} = k(k+1), \underline{\lambda} = \lambda - 1)$$

| Parabolas | | Point | Abscissa of the point |
|---|---|---|---|
| $f_{1,\lambda}$ | $f_{2,\lambda}$ | $\alpha_{12}$ | $\frac{1}{2}\left(\underline{k}\lambda + \sqrt{(\underline{k}\lambda + 2)^2 + 2(\underline{k}\underline{\lambda}^2 + 2\hat{\underline{k}} - 3)} + 2\right)$ |
| $f_{2,\lambda}$ | $f_{3,\lambda}$ | $\alpha_{23}$ | $\frac{1}{2}\left(\underline{k}\underline{\lambda} + \sqrt{(\underline{k}\underline{\lambda})^2 + 2(\underline{k}\underline{\lambda}^2 + 2\hat{\overline{k}} - 1)}\right)$ |
| $f_{3,\lambda}$ | $f_{4,\lambda}$ | $\alpha_{34}$ | $\frac{1}{2}\left(\underline{k}\lambda + \sqrt{(\underline{k}\lambda)^2 + 2(\underline{k}\lambda^2 + 2k^2)}\right)$ |
| $f_{4,\lambda}$ | $f_{1,\lambda}$ | $\alpha_{41}$ | $\frac{1}{2}\left(\underline{k}\lambda + \overline{k} \pm \sqrt{(\underline{k}\lambda + \overline{k})^2 + 2(\underline{k}\lambda^2 + 2\hat{\underline{k}} - \overline{k} - 1)}\right)$ |

# General Case & DCG



*Specifications of the parabolas* $\{f_{i,\lambda} \mid i = 1, 2, 3, 4\}$.

| Parabola | Axis | Directrix | Length of Latus Rectum | Vertex | Focus |
|---|---|---|---|---|---|
| $f_{1,\lambda}$ | $x = 1$ | $\underline{k}\, y = 3/4$ | $\underline{k}$ | $\left(1, (\hat{\underline{k}} + 1)/\underline{k}\right)$ | $\left(1, (8\hat{\bar{k}} + 5)/(4\underline{k})\right)$ |
| $f_{2,\lambda}$ | $x = -\underline{\lambda}$ | $\bar{k}\, y = 3/4$ | $\bar{k}$ | $\left(-\underline{\lambda}, (\hat{\bar{k}} + 1)/\bar{k}\right)$ | $\left(-\underline{\lambda}, (8\hat{\underline{k}} + 5)/(4\bar{k})\right)$ |
| $f_{3,\lambda}$ | $x = 0$ | $\underline{k}\, y = -1/4$ | $\underline{k}$ | $\left(0, (\hat{\underline{k}})/\underline{k}\right)$ | $\left(0, (8\hat{\bar{k}} + 1)/(4\underline{k})\right)$ |
| $f_{4,\lambda}$ | $x = -\lambda$ | $\bar{k}\, y = -1/4$ | $\bar{k}$ | $\left(-\lambda, \hat{\bar{k}}/\bar{k}\right)$ | $\left(-\lambda, (8\hat{\underline{k}} + 1)/(4\bar{k})\right)$ |

# General Case & DCG



*Specifications of the parabolas $\{f_{i,\lambda} \mid i = 1, 2, 3, 4\}$.*

POINTS OF INTERSECTION (IN $\mathbb{R}^2$) AMONG THE PARABOLAS $\{f_{i,\lambda} : i = 1, 2, 3, 4\}$ DEFINING $\mathcal{R}_{ak}$.

To obtain the value of $\{\alpha_{ij} \mid j = (i \bmod 4) + 1, i = 1, 2, 3, 4\}$, we have solved the following quadratic equations in $a$. Out of the two values of $a$ obtained, say $a = C \pm \sqrt{D}$, we define $\alpha$ as $C + \sqrt{D}$.

$$\begin{aligned}
\alpha_{23}: \quad & \frac{(a+\lambda-1)^2+k(k+1)+1}{2k+1} = \frac{a^2+k(k-1)}{2k-1} \\
& \text{or, } (2k-1)(a^2+2(\lambda-1)a+(\lambda-1)^2+k(k+1)+1) = (2k+1)(a^2+k(k-1)) \\
& \text{or, } 2a^2 - 2(2k-1)(\lambda-1)a - (2k-1)(\lambda-1)^2 - 2k^2 - 2k + 1 = 0 \\
& \text{or, } a = \tfrac{1}{2}\left((2k-1)(\lambda-1) \pm \sqrt{(2k-1)^2(\lambda-1)^2 + 2((2k-1)(\lambda-1)^2 + 2k^2 + 2k - 1)}\right) \\
& \text{or, } \alpha_{23} = \tfrac{1}{2}\left((2k-1)(\lambda-1) + \sqrt{(2k-1)^2(\lambda-1)^2 + 2((2k-1)(\lambda-1)^2 + 2k^2 + 2k - 1)}\right).
\end{aligned}$$

$$\begin{aligned}
\alpha_{12}: \quad & \frac{(a-1)^2+k(k-1)+1}{2k-1} = \frac{(a+\lambda-1)^2+k(k+1)+1}{2k+1} \\
& \text{or, } (2k+1)((a-1)^2+k(k-1)+1) = (2k-1)((a+\lambda-1)^2+k(k+1)+1) \\
& \text{or, } 2a^2 - 2((2k-1)\lambda+2)a - (2k-1)(\lambda-1)^2 - 2k^2 + 2k + 3 = 0 \\
& \text{or, } a = \tfrac{1}{2}\left((2k-1)\lambda + 2 \pm \sqrt{((2k-1)\lambda+2)^2 + 2((2k-1)(\lambda-1)^2 + 2k^2 - 2k - 3)}\right) \\
& \text{or, } \alpha_{12} = \tfrac{1}{2}\left((2k-1)\lambda + 2 + \sqrt{((2k-1)\lambda+2)^2 + 2((2k-1)(\lambda-1)^2 + 2k^2 - 2k - 3)}\right).
\end{aligned}$$

# General Case & DCG



$\alpha_{41}$: $\frac{(a+\lambda)^2+k(k+1)}{2k+1} = \frac{(a-1)^2+k(k-1)+1}{2k-1}$
or, $(2k-1)((a+\lambda)^2+k(k+1)) = (2k+1)((a-1)^2+k(k-1)+1)$
or, $2a^2 - 2(2k(1+\lambda)-\lambda+1)a - (2k-1)\lambda^2 - 2k^2 + 4k + 2 = 0$
or, $a = \frac{1}{2}\left((2k-1)\lambda + 2k + 1 \pm \sqrt{((2k-1)\lambda+2k+1)^2 + 2((2k-1)\lambda^2 + 2k^2 - 4k - 2)}\right)$
or, $\alpha_{41} = \frac{1}{2}\left((2k-1)\lambda + 2k + 1 + \sqrt{((2k-1)\lambda+2k+1)^2 + 2((2k-1)\lambda^2 + 2k^2 - 4k - 2)}\right)$.

$\alpha_{34}$: $\frac{a^2+k(k-1)}{2k-1} = \frac{(a+\lambda)^2+k(k+1)}{2k+1}$
or, $(2k+1)(a^2+k(k-1)) = (2k-1)((a+\lambda)^2+k(k+1))$
or, $2a^2 - 2(2k-1)\lambda - (2k-1)\lambda^2 - 2k^2 = 0$
or, $a = \frac{1}{2}\left((2k-1)\lambda \pm \sqrt{(2k-1)^2\lambda^2 + 2((2k-1)\lambda^2 + 2k^2)}\right)$
or, $\alpha_{34} = \frac{1}{2}\left((2k-1)\lambda + \sqrt{(2k-1)^2\lambda^2 + 2((2k-1)\lambda^2 + 2k^2)}\right)$.

# Algorithm DCG (1)

$n_{\max} \leftarrow 0$

**for** $k' \leftarrow k_{\min}$ to $k_{\max}$

$\Lambda \leftarrow S[0], i \leftarrow 0$

FIND-PARAMS$(A, \Lambda, k')$

**while** $i < m$ **and** $n_{\max} < n$ ▷ for all $a$'s of first run

$[s', s''] \leftarrow [r', r''] \leftarrow [A[i][1], A[i][2]]$

$\Lambda \leftarrow A[i][0] + S[0], j \leftarrow 1$

**while** $j < n$ **and** $s'' \geqslant r'$ **and** $s' \leqslant r''$ ▷ verifying other $n-1$ runs

$\Lambda \leftarrow \Lambda + S[j], k \leftarrow k' + j$

$s' \leftarrow \left\lceil \frac{(\Lambda-1)^2+k(k+1)+1}{2k+1} \right\rceil, s'' \leftarrow \left\lfloor \frac{\Lambda^2+k(k+1)}{2k+1} \right\rfloor$

**if** $s'' \geqslant r'$ **and** $s' \leqslant r''$

$[r', r''] \leftarrow [\max(r', s'), \min(r'', s'')]$

**if** $n_{\max} < j$

$n_{\max} \leftarrow j, k_{\mathrm{off}} \leftarrow k', [r_{\min}, r_{\max}] \leftarrow [r', r'']$

**print** "$S$ is circular for $n_{\max}$ runs; starting run $= k_{\mathrm{off}}$; $r \in [r_{\min}, r_{\max}]$."

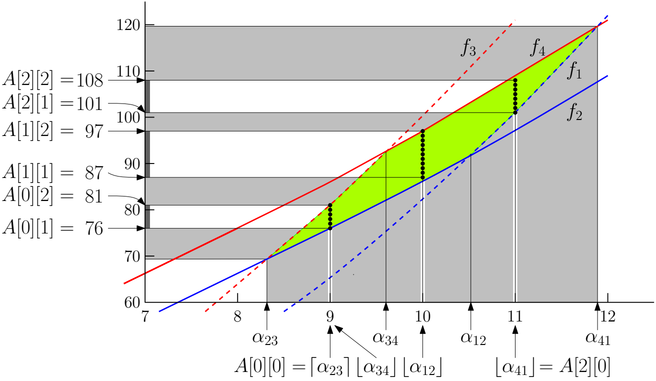
# Algorithm DCG



## *Procedure* FIND-PARAMS

Compute $\{\alpha_{uv} \mid 1 \leqslant u \leqslant 4 \ \wedge v = (u+1) \bmod 4\}$ ▷ (from Tables)
$i \leftarrow 0$
**for** $a \leftarrow \lceil \alpha_{23} \rceil$ to $\lfloor \alpha_{41} \rfloor$
    $A[i][0] \leftarrow a$ ▷ computing $r'$
    **if** $a < \alpha_{12}$
        $A[i][1] \leftarrow \lceil f_{2,\lambda}(a,k) \rceil$
    **else**
        $A[i][1] \leftarrow \lceil f_{1,\lambda}(a,k) \rceil$ ▷ computing $r''$
    **if** $a < \alpha_{34}$
        $A[i][2] \leftarrow \lfloor f_{3,\lambda}(a,k) \rfloor$
    **else**
        $A[i][2] \leftarrow \lfloor f_{4,\lambda}(a,k) \rfloor$
    $i \leftarrow i+1$
$m \leftarrow i$

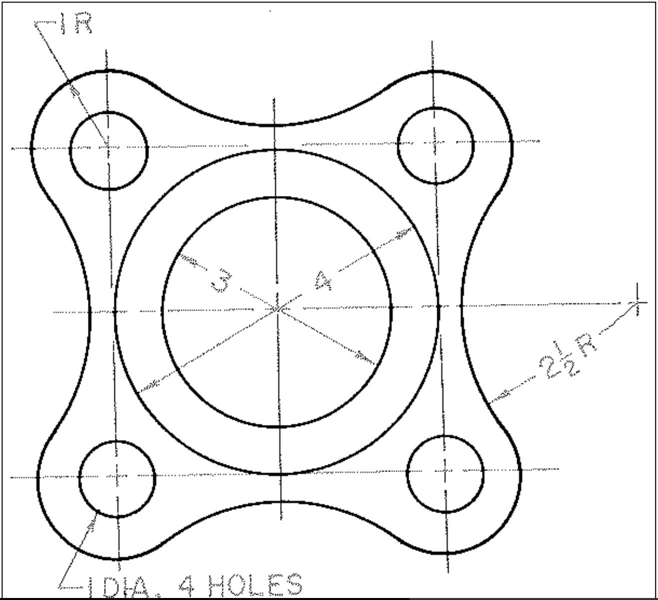
# Algorithm DCG (3)



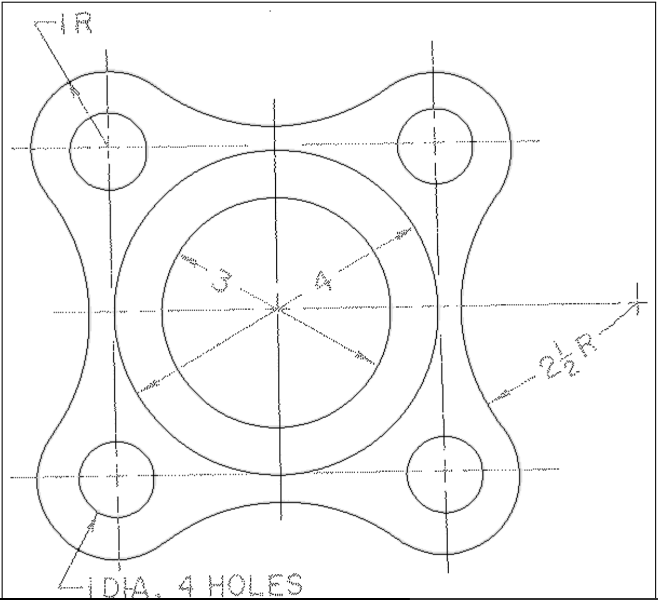

FIND-PARAMS on a run-length 7:

Solution space $\mathcal{R}_{ak}$ of the radius intervals $\{[r'_j, r''_j] \mid j = 0, 1, 2\}$ corresponding to $m = 3$ square numbers lying in $\left[\lceil \alpha_{23} \rceil^2, \lfloor \alpha_{41} \rfloor^2\right] = [9^2, 11^2]$.

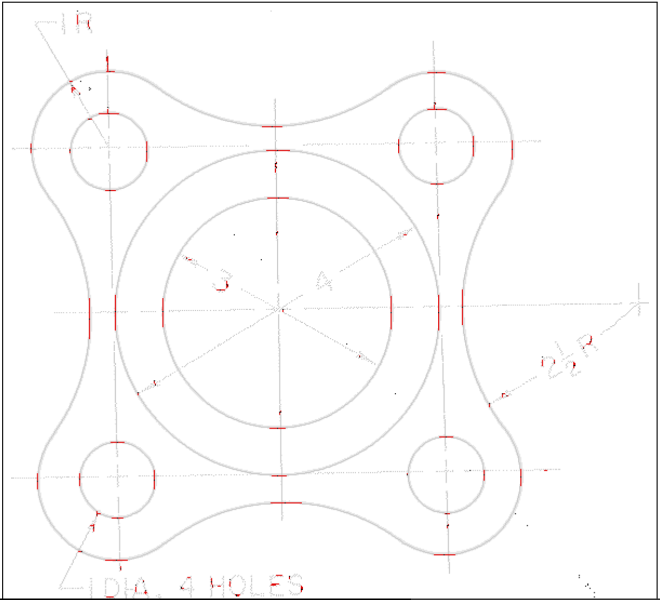
# Arc Segmentation



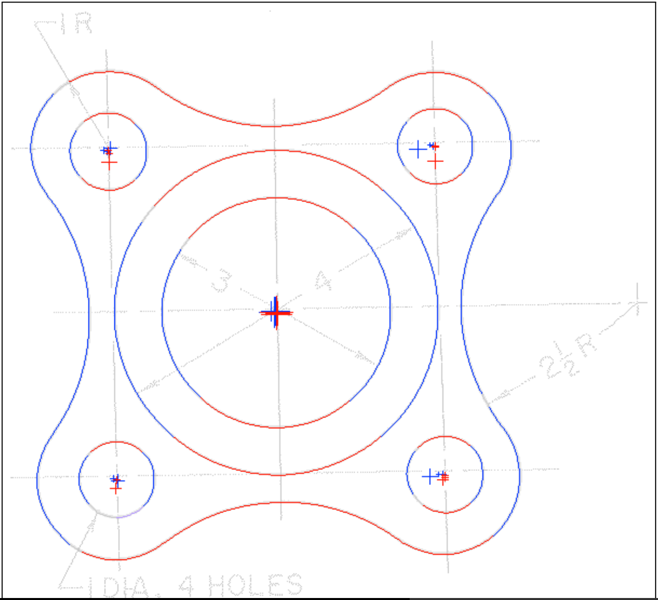
# Arc Segmentation



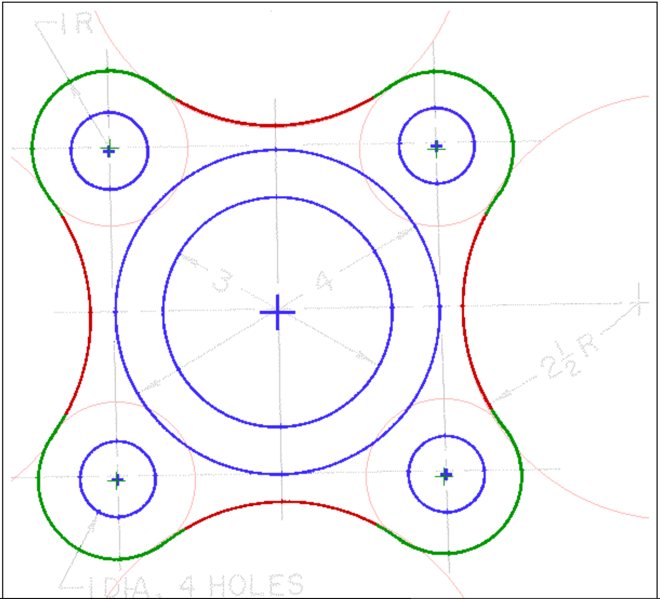
# Arc Segmentation



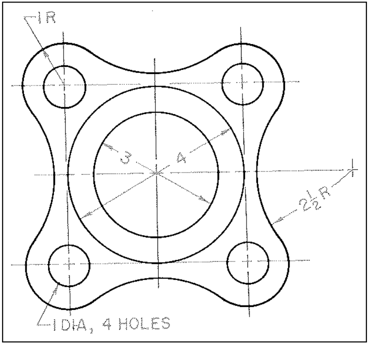
# Arc Segmentation



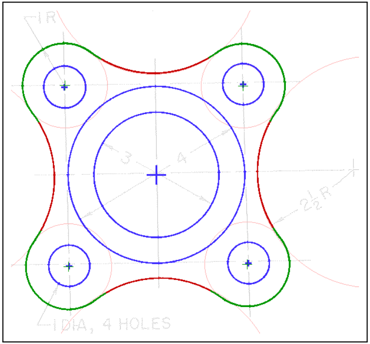
# Arc Segmentation



# Arc Segmentation




input




output

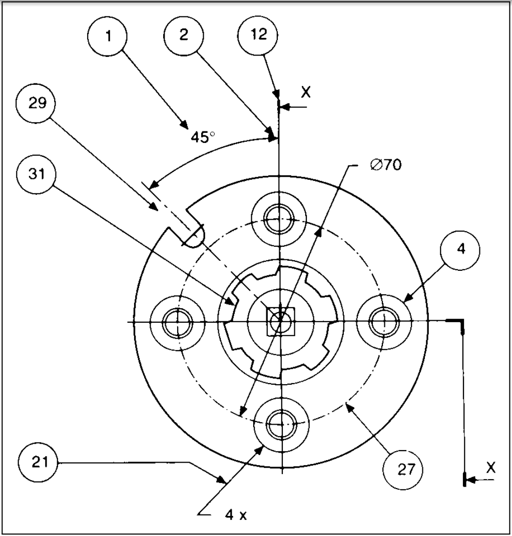
# Arc Segmentation

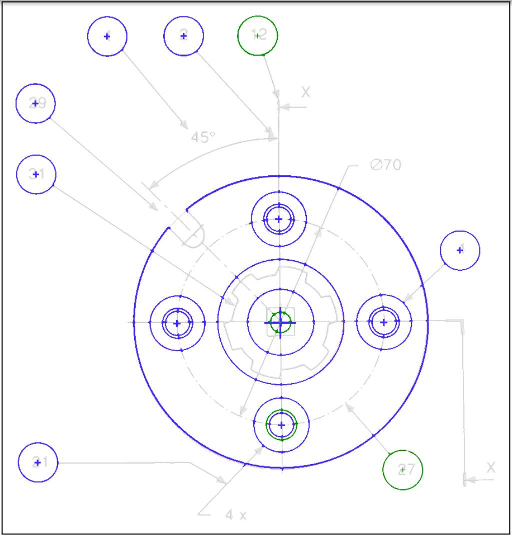
| Algorithm | Inventors | Year |
|---|---|---|
| *Hough transform* | Davies **1984**, Illingworth & Kittler **1988**, Yip et al. **1992**, Chen & Chung **2001**, Kim & Kim **2005**, Chiu & Liaw **2005**,... | |
| *Voronoi diagram* | Coeurjolly *et al.* | 2004 |
| *Chord & Sagitta* | Bera, Bhowmick & Bhattacharya | 2010 |
| *Discrete Curvature*[a] | Pal, Dutta & Bhowmick | 2012 |
| *Number Theory*[b] | Pal & Bhowmick | 2012 |
| *Number Theory & Graph Theory*[c] | Bhowmick & Pal | 2014 |

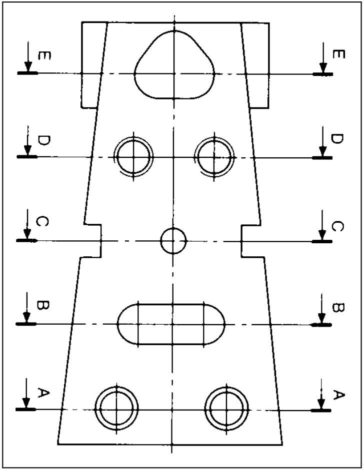
[a] S. Pal, R. Dutta & P. Bhowmick, Circular Arc Segmentation by Curvature Estimation and Geometric Validation, *Intl. Journal Image & Graphics*, **12**:24p, 2012.

[b] S. Pal & P. Bhowmick, Determining Digital Circularity Using Integer Intervals, *Journal of Mathematical Imaging & Vision*, **42**(1):1-24, 2012.

[c] S. Pal & P. Bhowmick, Fast Circular Arc Segmentation Based on Approximate Circularity and Cuboid Graph, *Journal of Mathematical Imaging & Vision*, **49**:98-122, 2014.


1
2
12
X
29
45°
31
∅70
4
21
27
X
4 x


45°
X
∅70
4 x
X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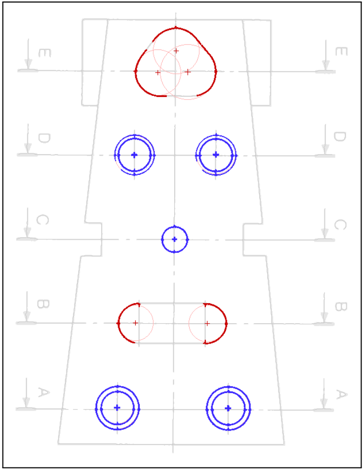
E
D
D
C
C
B
B
A
A


E
E
D
D
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C
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A
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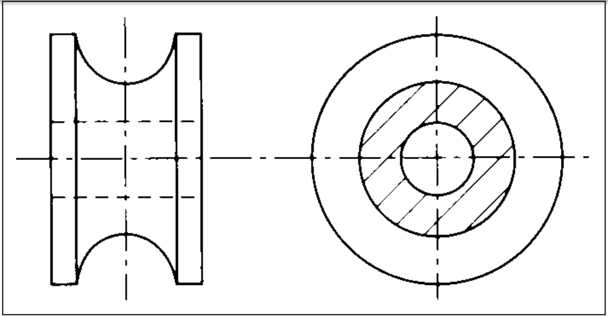
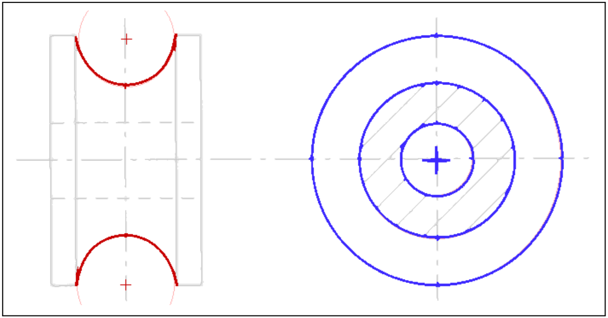
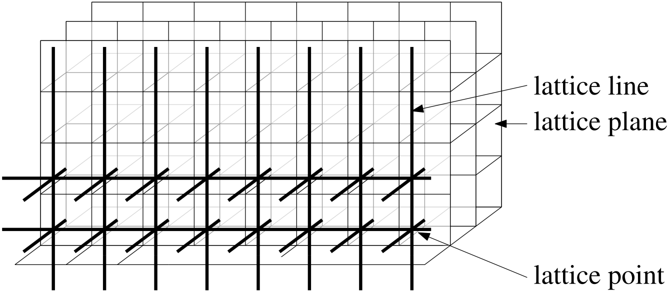




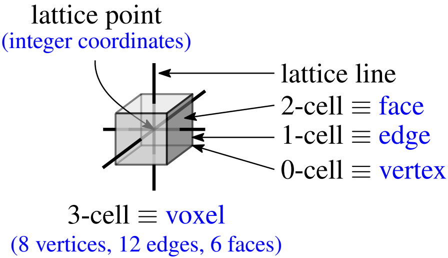
# Discretization of Sphere

# Lattice, cells, voxels, adjacency (1)



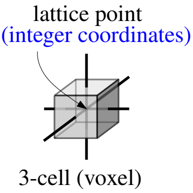
# Lattice, cells, voxels, adjacency

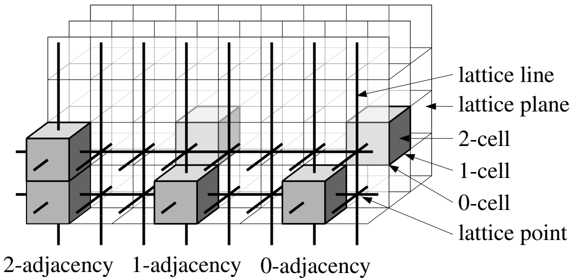




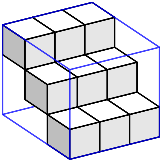
# Lattice, cells, voxels, adjacency





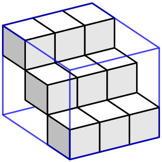



# Discretization Models *(general surface)*

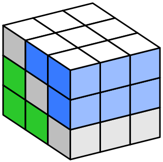


*naive*

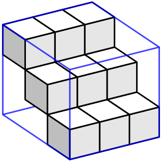
# Discretization Models *(general surface)*



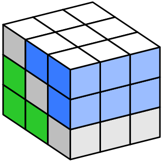
*naive*

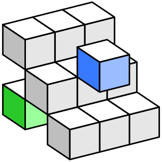


# Discretization Models *(general surface)*



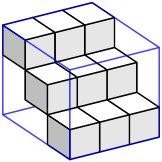
*naive*



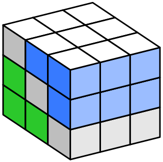


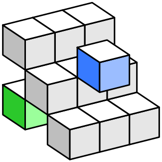
no 2-path

# Discretization Models *(general surface)*

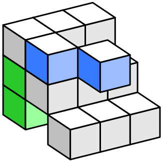


*naive*



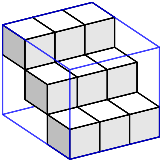


no 2-path



1-path

# Discretization Models *(general surface)*



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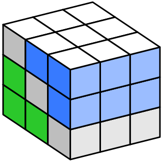
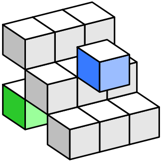
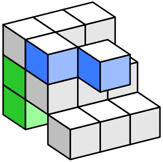
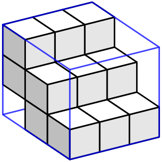




no 2-path

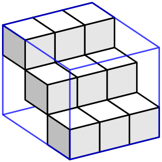


1-path

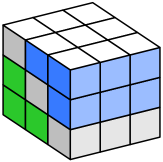


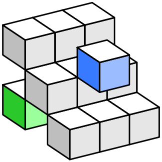
*standard*

# Discretization Models *(general surface)*

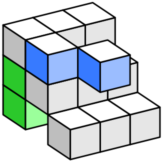


*naive*

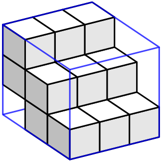




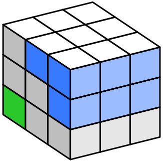
no 2-path



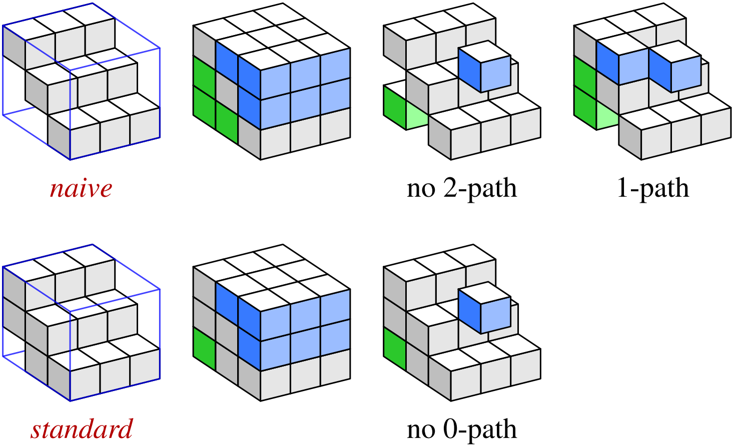
1-path



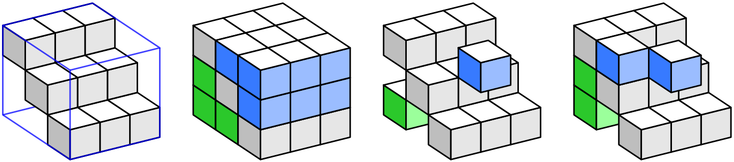
*standard*



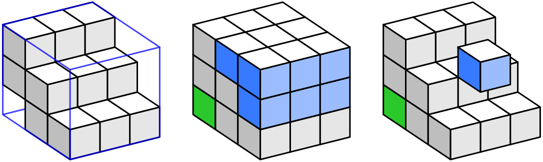
# Discretization Models *(general surface)*



# Discretization Models *(general surface)*



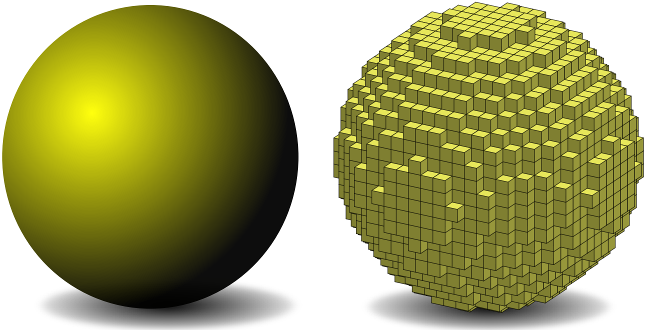
*naive* no 2-path 1-path



*standard* no 0-path

*Naive = 2-minimal. Standard = 0-minimal.*

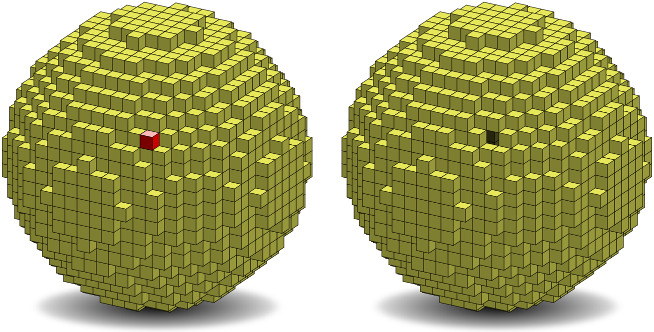
# Naive Sphere



**Problem Statement:** Given integer radius and integer center[1], construct the *naive sphere* whose *every voxel* is *non-redundant* and *lies as much close as possible to the real sphere*.

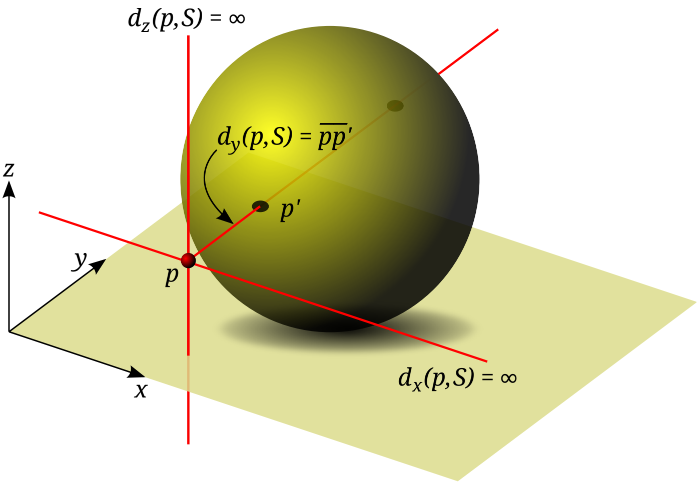
[1] w.l.o.g., center $= (0, 0, 0)$

# Non-redundant



# Isothetic distance

To formalize “*as much close as possible to the real sphere*”, we define $d_\perp(p,S) = \min\{d_x(p,S), d_y(p,S), d_z(p,S)\}$.



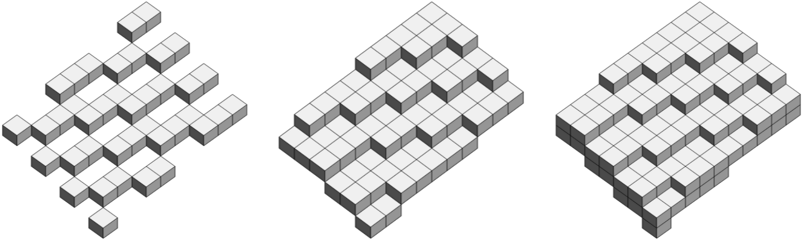
# Discretization Models *(plane & sphere)* (1)

*Real plane* $\Pi(a,b,c,\mu) : ax+by+cz=\mu$.

*Digital plane*

$\Pi^{\mathbb{Z}}(a,b,c,\mu,\omega) = \left\{(i,j,k)\in\mathbb{Z}^3 : \mu-\frac{\omega}{2}\leqslant ai+bj+ck<\mu+\frac{\omega}{2}\right\}$,
which is of *thickness* $\omega$ and centered on $\Pi$.

**Example:** $6x+13y+27z=0$

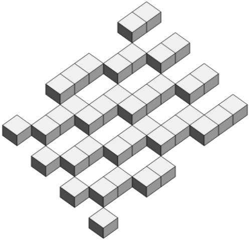


$\omega=15$ $\omega=27$ $\omega=46$

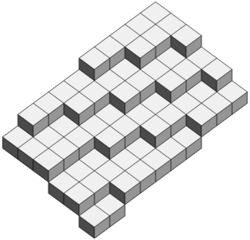
# Discretization Models *(plane & sphere)* (2)

| under-digitized | naive | standard |
|---|---|---|
| $\omega < \max(\lvert a\rvert, \lvert b\rvert, \lvert c\rvert)$ | $\omega = \max(\lvert a\rvert, \lvert b\rvert, \lvert c\rvert)$ $\Leftrightarrow$ 2-*minimal* | $\omega = \lvert a\rvert + \lvert b\rvert + \lvert c\rvert$ $\Leftrightarrow$ 0-*minimal* |

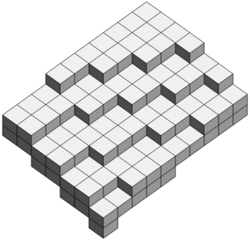
Example: $6x + 13y + 27z = 0$



$\omega = 15$



$\omega = 27$



$\omega = 46$

# Discretization Models *(plane & sphere)* (3)

**Lemma**

*For a point* $p = (i, j, k)$ *and a real plane* $\Pi : ax + by + cz = 0$,

$$d_\perp(p, \Pi) = \frac{|ai + bj + ck|}{\max(|a|, |b|, |c|)}.$$

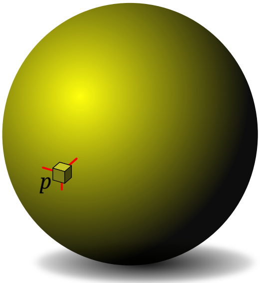
**Theorem (Point-to-Plane Distance[a])**

$$d_\perp(p, \Pi_r^{\mathbb{R}}(s,t)) \leqslant \begin{cases} \frac{1}{2} & \forall\, p \in \Pi_1^{\mathbb{Z}}(s,t) \quad \leftarrow \textit{naive} \\ \frac{3}{2} & \forall\, p \in \Pi_2^{\mathbb{Z}}(s,t) \quad \leftarrow \textit{standard} \end{cases}$$

[a] R. Biswas and P. Bhowmick, On different topological classes of spherical geodesic paths and circles in $\mathbb{Z}^3$, *Theoretical Computer Science* **605**:146–163, 2015.

# Discretization Models *(plane & sphere)*





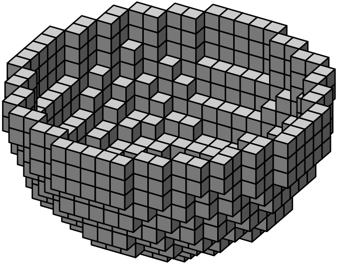

$$p = (i, j, k) \in \mathbb{Z}^3,$$
$$X = \{|i|, |j|, |k|\},$$
$$h = |i| + |j| + |k|,$$
$$s = i^2 + j^2 + k^2.$$

## Theorem (Naive & Standard Spheres)

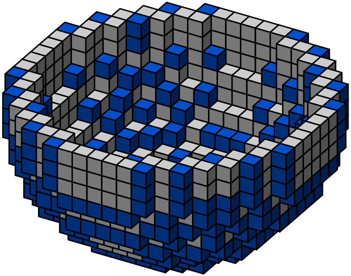
$$\mathbb{S}_1 = \left\{ p \in \mathbb{Z}^3 : \begin{array}{l} \left(r^2 - \max(X) \leqslant s < r^2 + \max(X)\right) \\ \wedge \left(\left(s \neq r^2 + \max(X) - 1\right) \vee \left(\mathrm{mid}(X) \neq \max(X)\right)\right) \end{array} \right\}.$$

$$\mathbb{S}_2 = \left\{ p \in \mathbb{Z}^3 : r^2 - h \leqslant s < r^2 + h \right\}.$$

# Discretization Models *(plane & sphere)* (5)



*naive*

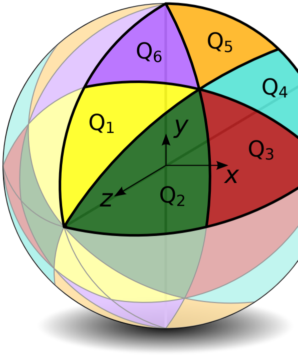


*standard*

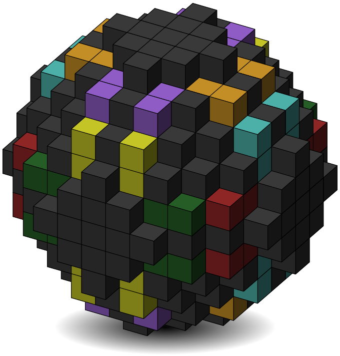
**Theorem (Point-to-Sphere Distance)**

$$d_\perp(p, S) \leqslant \begin{cases} \frac{1}{2} & \forall\, p \in \mathbb{S}_1 \quad \leftarrow \textit{naive} \\ 2 & \forall\, p \in \mathbb{S}_2 \quad \leftarrow \textit{standard} \end{cases}$$

# Symmetry *(quadraginta octants)*

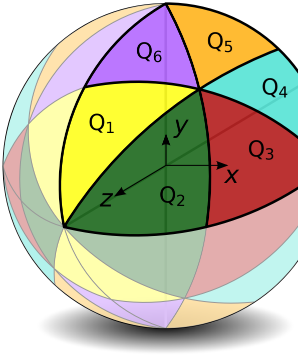



$r = 5$

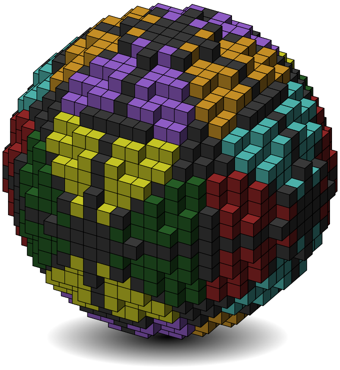


#q-octants = #permutations of $(\pm x, \pm y, \pm z) = 3! \times 2^3 = 48$.

# Symmetry *(quadraginta octants)*



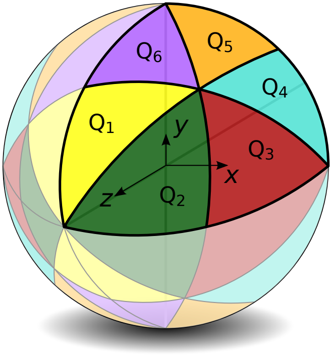

$r = 10$

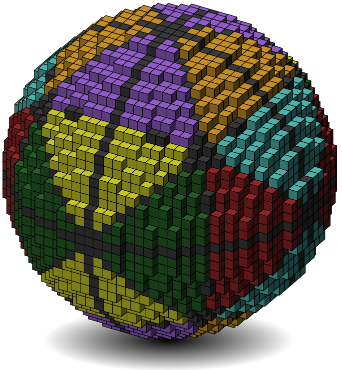


#q-octants = #permutations of $(\pm x, \pm y, \pm z) = 3! \times 2^3 = 48$.

# Symmetry *(quadraginta octants)*

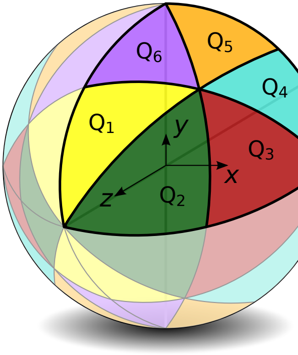
$r = 15$



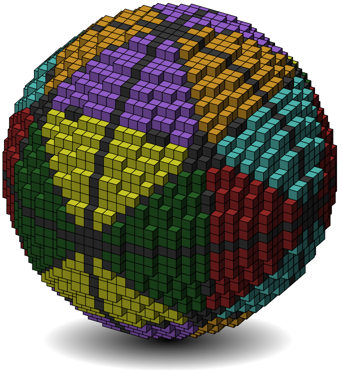



#q-octants = #permutations of $(\pm x, \pm y, \pm z) = 3! \times 2^3 = 48$.

# Symmetry *(quadraginta octants)*

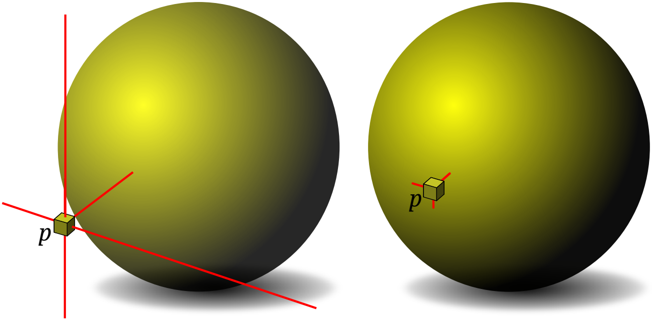



$r = 15$



#q-octants = #permutations of $(\pm x, \pm y, \pm z) = 3! \times 2^3 = 48.$

# Distance bound

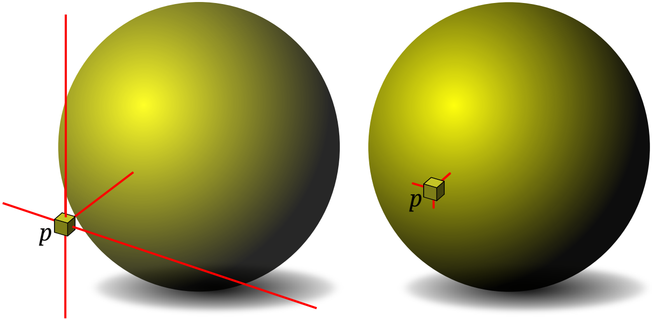



Here $p = (i, j, k) \in \mathbb{Z}^3$, $s = i^2 + j^2 + k^2$, $X = \{|i|, |j|, |k|\}$.

**Lemma ($\mathbb{S}_1$-to-$S$ Distance)**

$$p \in \mathbb{S}_1 \implies d_\perp(p, S) = \left| k - \sqrt{r^2 - (i^2 + j^2)} \right|.$$

# Distance bound

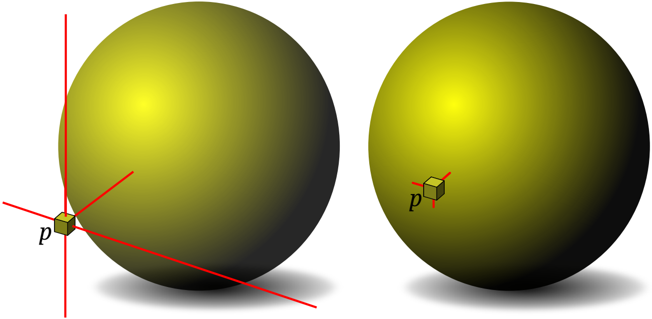



Here $p = (i, j, k) \in \mathbb{Z}^3, s = i^2 + j^2 + k^2, X = \{|i|, |j|, |k|\}$.

**Lemma (Supremum Distance)**

$$p \in \mathbb{S}_1 \implies d_\perp(p, S) \lneqq \frac{1}{2}.$$

# Properties & Characterization

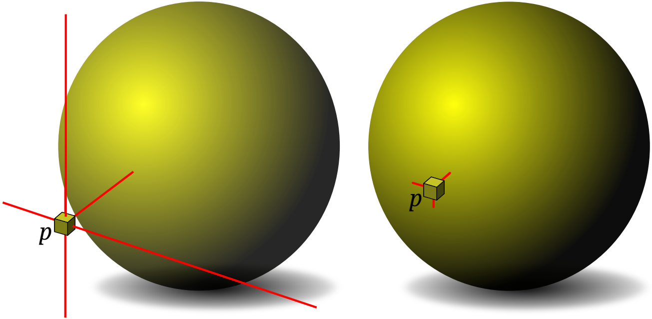



Here $p = (i, j, k) \in \mathbb{Z}^3, s = i^2 + j^2 + k^2, X = \{|i|, |j|, |k|\}$.

**Lemma (Square Sum Interval)**

$$p \in \mathbb{S}_1 \implies s \in [r^2 - k, r^2 + k - 1].$$

# Properties & Characterization

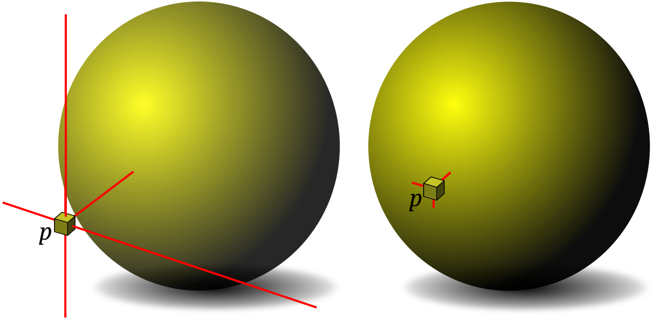



Here $p = (i, j, k) \in \mathbb{Z}^3, s = i^2 + j^2 + k^2, X = \{|i|, |j|, |k|\}$.

## Theorem (Simple Voxel)

$p : d_\perp(p, S) < \frac{1}{2}$ *is simple/redundant*
$\Leftrightarrow \left(s = r^2 + \max(X) - 1\right) \wedge \left(\mathrm{mid}(X) = \max(X)\right)$,
*where* mid *denotes the median.*

# Properties & Characterization




Here $p = (i, j, k) \in \mathbb{Z}^3, s = i^2 + j^2 + k^2, X = \{|i|, |j|, |k|\}$.

**Theorem (Lattice Sphere)**

$$\mathbb{S}_1 = \left\{ p \in \mathbb{Z}^3 : \left(r^2 - \max(X) \leqslant s < r^2 + \max(X)\right) \wedge \left(\left(s \neq r^2 + \max(X) - 1\right) \vee \left(\mathrm{mid}(X) \neq \max(X)\right)\right) \right\}.$$

# Number-theoretic Properties (1)

**Lemma (Interval)**

*The interval* $I_n = [(2n-1)r - n(n-1), (2n+1)r - n(n+1) - 1]$ *contains the sum of the squares of x- and y-coordinates of the voxels of* $\mathbb{S}_1$ *whose z-coordinates are* $r - n$*, for* $n \geqslant 1$.

**Lemma (Interval Length)**

*The lengths of the intervals* $I_n$*, starting from* $I_1$*, decrease constantly by 2.*

# Number-theoretic Properties (2)

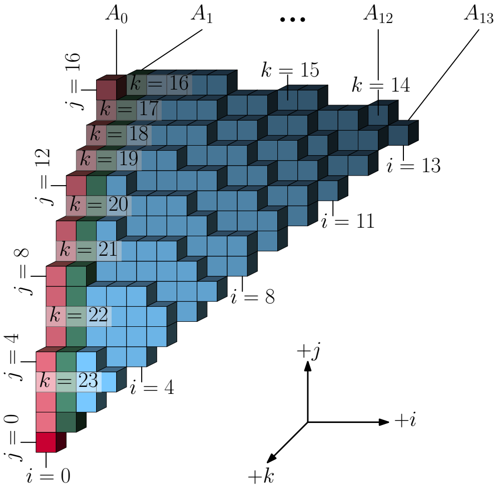
## Theorem (Interval Recurrence)

*The sum of squares of x- and y-coordinates of voxels lying on $\mathbb{S}_1$ and having z-coordinate $r-n$, lies in $I_n := [u_n, v_n := u_n + l_n - 1]$, where*

$$u_n = \begin{cases} 0 & \textit{if } n = 0 \\ u_{n-1} + l_{n-1} & \textit{otherwise;} \end{cases} \tag{1}$$

$$l_n = \begin{cases} r & \textit{if } n = 0 \\ 2r-2 & \textit{if } n = 1 \\ l_{n-1} - 2 & \textit{otherwise.} \end{cases} \tag{2}$$

# Number-theoretic Properties (3)




First q-octant of the naive sphere of $r = 23$.

# Number-theoretic Properties



## Theorem (Next Voxel)

*If $(i, j, k)$ is the current voxel of $A_i$, then the next voxel of $A_i$ is $(i, j+1, k-d)$, where $d \in \{\varnothing, 0, 1\}$ is given as follows.*

| *Interval* | $j < k-1$ | $j = k-1$ |
|---|---|---|
| $K_0$ | 0 | $\varnothing$ |
| $K_{-1}$ | 1 | $\varnothing$ |

*Here, $d = \varnothing$ implies that there does not exist an appropriate value of $d$.*

# Number-theoretic Properties (5)

## Theorem (Next Arc)

*If $(i, j = i, k)$ is the first voxel of $A_i$, then the first voxel of $A_{i+1}$ is $(i+1, i+1, k-d)$, where $d \in \{\varnothing, 0, 1, 2\}$ is given as follows.*

| *Interval* | $i < k-2$ | $i = k-2$ | $i = k-1$ |
|---|---|---|---|
| $K_0$ | 0 | 0 | $\varnothing$ |
| $K_{-1}$ | 1 | $\varnothing, 1$ | $\varnothing$ |
| $K_{-2}$ | 2 | $\varnothing$ | $\varnothing$ |

# Algorithm LS3



**Algorithm 1:** LS3 $(r)$

int $i \leftarrow j \leftarrow 0, k \leftarrow k_0 \leftarrow r, s \leftarrow s_0 \leftarrow 0, v \leftarrow v_0 \leftarrow r-1, l \leftarrow l_0 \leftarrow 2v_0$

voxel set $S \leftarrow \{\,\}$

**while** $i \leqslant k$ **do** ▷ arc generator

  **while** $j \leqslant k$ **do** ▷ voxel generator

    **if** $s > v$ **then** ▷ $d = 1$ (Theorem 23)

      $k \leftarrow k-1, v \leftarrow v+l, l \leftarrow l-2$ ▷ Theorem 22

    **if** $((j \leqslant k) \wedge ((s \neq v) \vee (j \neq k))$ **then** ▷ Lattice Sphere Thm

      $S \leftarrow S \cup \{(i',j',k') : \{|i'|\} \cup \{|j'|\} \cup \{|k'|\} = \{i,j,k\}\}$

    $s \leftarrow s + 2j + 1, j \leftarrow j + 1$

  $s_0 \leftarrow s_0 + 4i + 2, i \leftarrow i + 1$

  **while** $(s_0 > v_0) \wedge (i \leqslant k_0)$ **do** ▷ next arc init (Theorem 24)

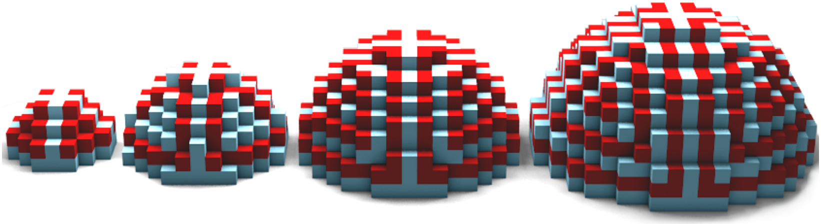
    $k_0 \leftarrow k_0 - 1, v_0 \leftarrow v_0 + l_0, l_0 \leftarrow l_0 - 2$ ▷ Theorem 22

  $j \leftarrow i, k \leftarrow k_0, v \leftarrow v_0, l \leftarrow l_0, s \leftarrow s_0$

**return** $S$

# Algorithm LS3





And so the lattice spheres are produced...

# Techniques

| Algorithm | Principle | PR | PL | IntOp |
|---|---|---|---|---|
| Montani-Scopigno, 1990 [7] | Incremental | No | No | Yes |
| Andres, 1994 [1] | Incremental | No | Yes | Yes |
| Andres-Jacob, 1997 [2] | Incremental | No | Yes | No |
| Roget-Sitaraman, 2013 [8] | Incremental | No | No | Yes |
| Toutant *et al.*, 2013 [9] | Morphology | No | No | No |
| Biswas-Bhowmick, 2015 [5] | *N.T.*[a] | No | No | Yes |
| Biswas-Bhowmick, 2015 [3] | *N.T.*[b] | Yes | Yes | Yes |

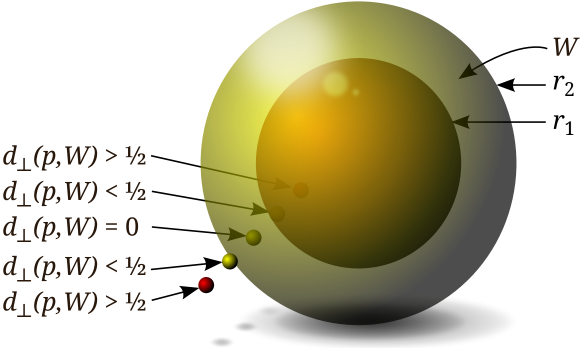
**PR** = print by run; **PL** = print by layer; **IntOp** = based on integer operations; *N.T.* = based on elementary number-theoretic properties.

[a] From Prima Quadraginta Octant to Lattice Sphere through Primitive Integer Operations, *Theoretical Computer Science* (in press), 2015 (doi: `http://dx.doi.org/10.1016/j.tcs.2015.11.018`)

[b] Layer the sphere, *The Visual Computer* **31**: 787–797, 2015

# Spherical Shell

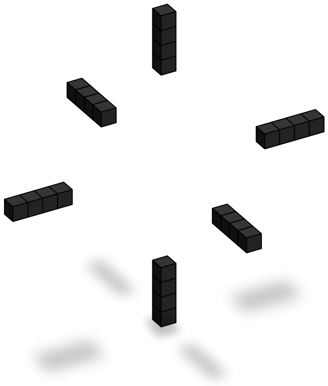





For a spherical shell $W$, the voxel set is

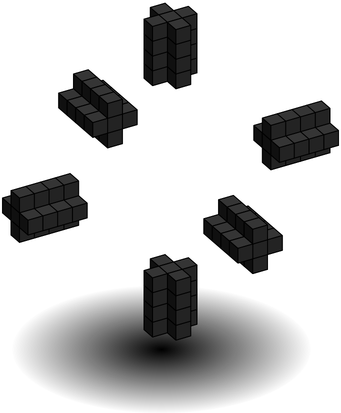
$$\mathbb{S} = \left\{p : 0 \leqslant d_\perp(p, W) < \frac{1}{2}\right\}.$$

# *Demo:* SPHEREBY48SYM$(r_1 = 7, r_2 = 10)$



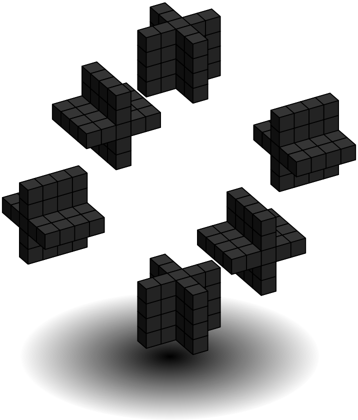
$i = 0,\ j = 0$

# *Demo:* SPHEREBY48SYM$(r_1 = 7, r_2 = 10)$



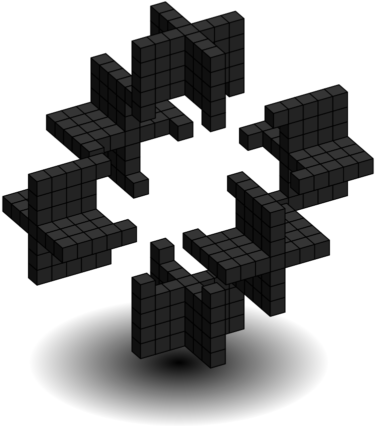
$i = 0,\ j = 1$

# *Demo:* SPHEREBY48SYM($r_1 = 7, r_2 = 10$)



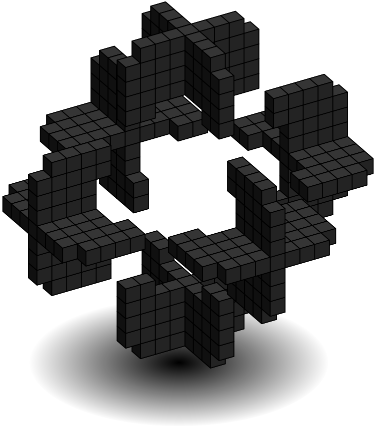
$i = 0,\ j = 2$

# *Demo:* SPHEREBY48SYM$(r_1 = 7, r_2 = 10)$



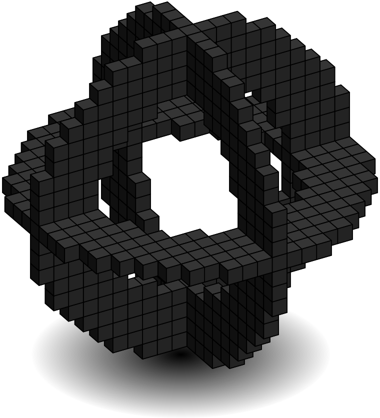
$i = 0, j = 3$

# *Demo:* SPHEREBY48SYM$(r_1 = 7, r_2 = 10)$



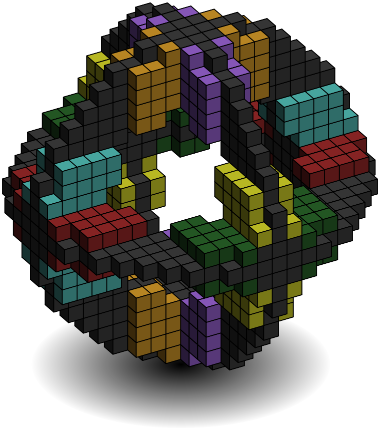
$i = 0, j = 4$

# *Demo:* SPHEREBY48SYM$(r_1 = 7, r_2 = 10)$



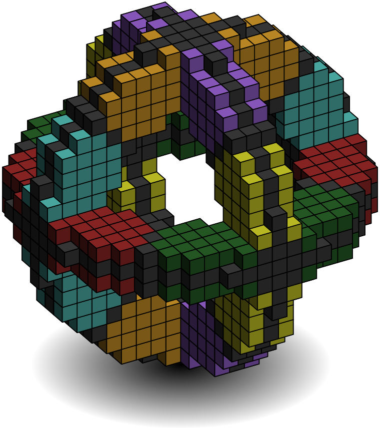
$i = 0,\ j = 7$

# *Demo:* SPHEREBY48SYM$(r_1 = 7, r_2 = 10)$



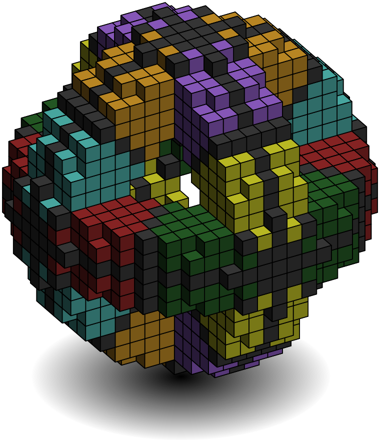
$i = 1, j = 4$

# *Demo:* SPHEREBY48SYM($r_1 = 7, r_2 = 10$)



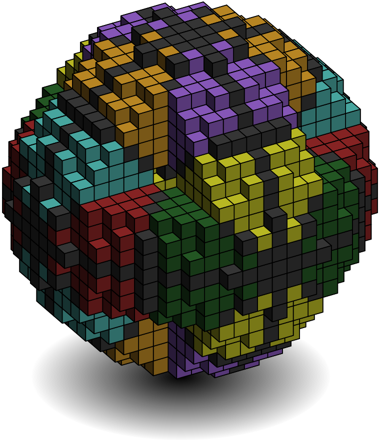
$i = 1, j = 7$

# *Demo:* SPHEREBY48SYM($r_1 = 7, r_2 = 10$)



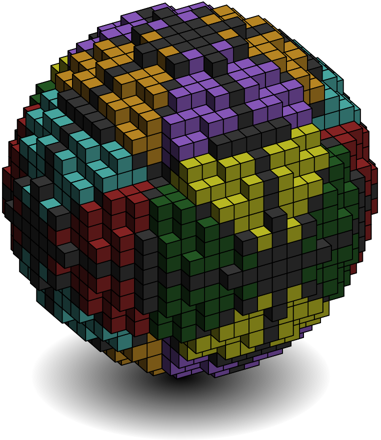
$i = 2, j = 7$

# *Demo:* SPHEREBY48SYM$(r_1 = 7, r_2 = 10)$



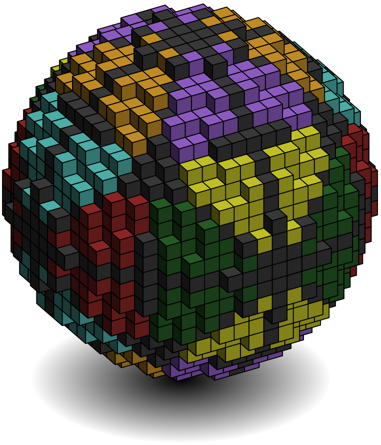
$i = 3, j = 6$

# *Demo:* SPHEREBY48SYM$(r_1 = 7, r_2 = 10)$



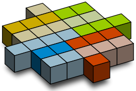
$i = 4, \ j = 6$

# *Demo:* SPHEREBY48SYM($r_1 = 7, r_2 = 10$)



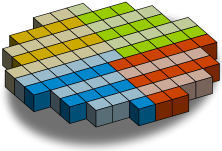
End

# *Demo:* LAYERTHESPHERE$(r_1 = 7, r_2 = 10)$



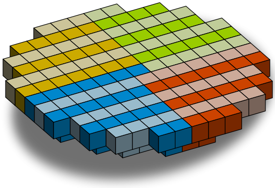
$k = -10$

# *Demo:* LAYERTHESPHERE$(r_1 = 7, r_2 = 10)$



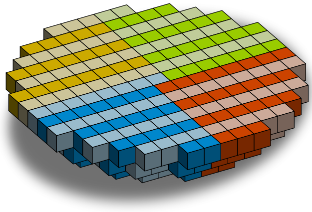
$$k = -9$$

# *Demo:* LAYERTHESPHERE$(r_1 = 7, r_2 = 10)$



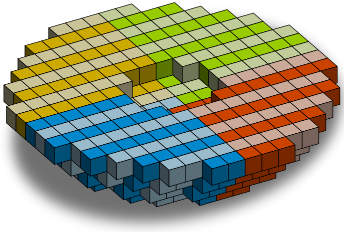
$k = -8$

# *Demo:* LAYERTHESPHERE$(r_1 = 7, r_2 = 10)$



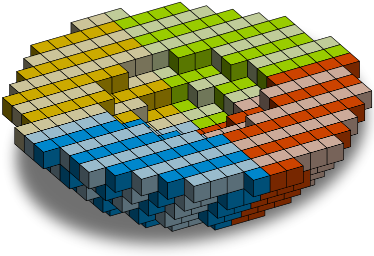
$k = -7$

# *Demo:* LayerTheSphere$(r_1 = 7, r_2 = 10)$



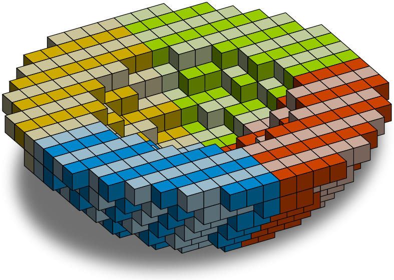
$$k = -6$$

# *Demo:* LAYERTHESPHERE$(r_1 = 7, r_2 = 10)$



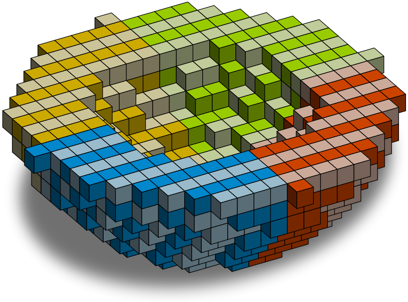
$$k = -5$$

# *Demo:* LAYERTHESPHERE$(r_1 = 7, r_2 = 10)$



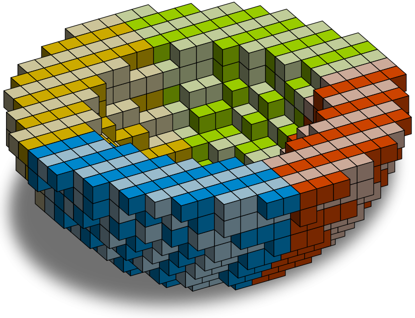
$k = -4$

# *Demo:* LAYERTHESPHERE$(r_1 = 7, r_2 = 10)$



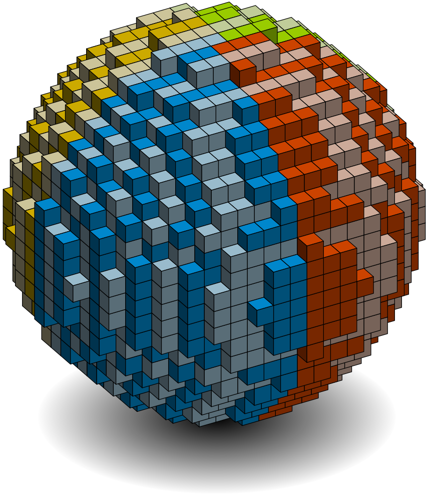
$k = -3$

# *Demo:* LAYERTHESPHERE$(r_1 = 7, r_2 = 10)$



$$k = -2$$

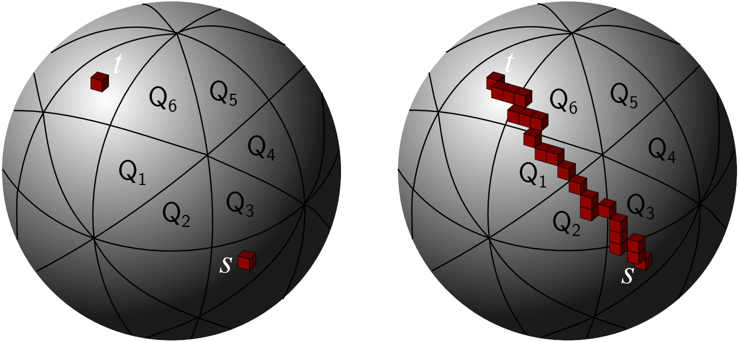
# *Demo:* LAYERTHESPHERE$(r_1 = 7, r_2 = 10)$



$k = 10$ (end)

# Spherical Geodesics

# Spherical Geodesics [4]




$r = 12,\ s = (10, -2, 6) \in Q_{15},\ t = (-3, 10, 6) \in Q_{12}.$

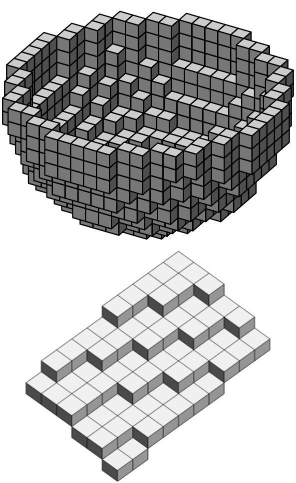
Naive sphere, Standard plane $\Longrightarrow$ *Class* NS ($l = 1$)[a]

[a] R. Biswas and P. Bhowmick, On different topological classes of spherical geodesic paths and circles in $\mathbb{Z}^3$, *Theoretical Computer Science* **605**:146–163, 2015.

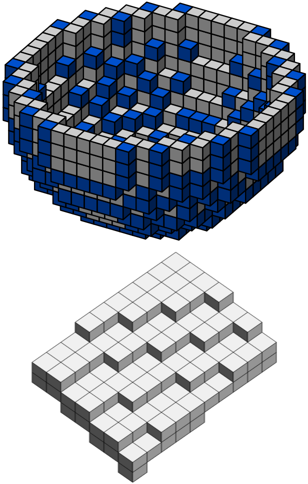
# DSGP Topological Classes



Naive model



Standard model



# DSGP Topological Classes



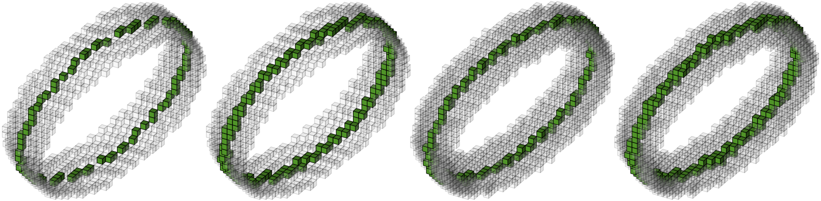
NN) Naive-naive $(m = 1, n = 1)$

NS) Naive-standard $(m = 1, n = 2)$

SN) Standard-naive $(m = 2, n = 1)$

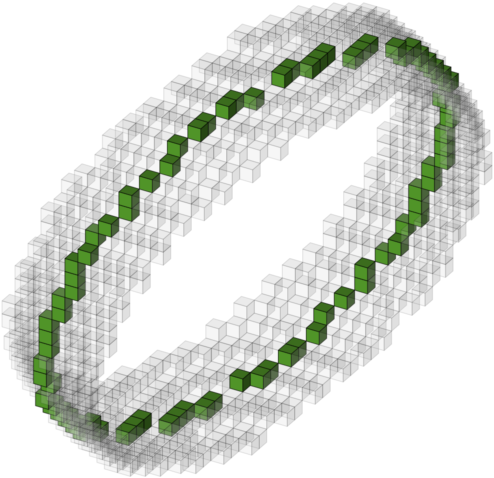
SS) Standard-standard $(m = 2, n = 2)$

**Example:** $r = 17$, $s = (-6, -1, 16)$, $t = (2, 14, 10)$.



NN $(1, 1)$ NS $(1, 2)$ SN $(2, 1)$ SS $(2, 2)$

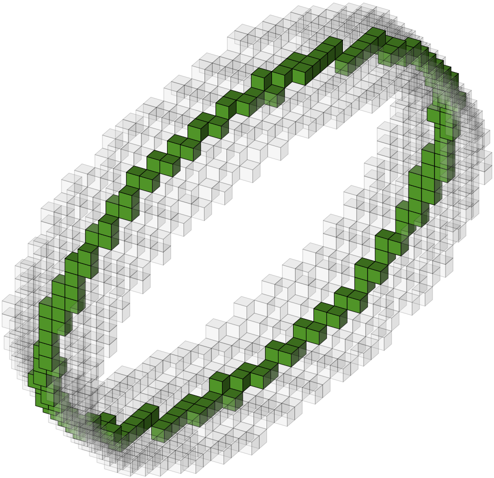
# DSGP Topological Classes (3)



NN $(1,1)$

# DSGP Topological Classes

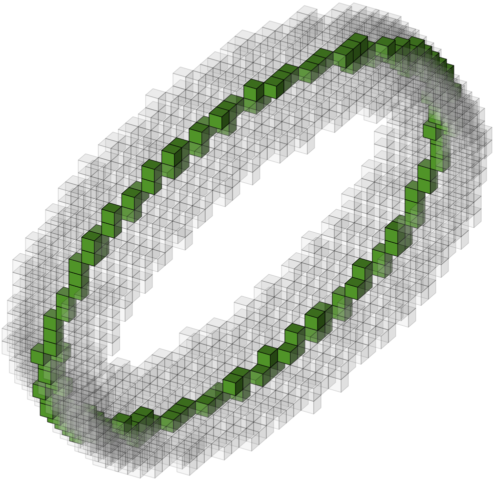




NS $(1, 2)$

# DSGP Topological Classes

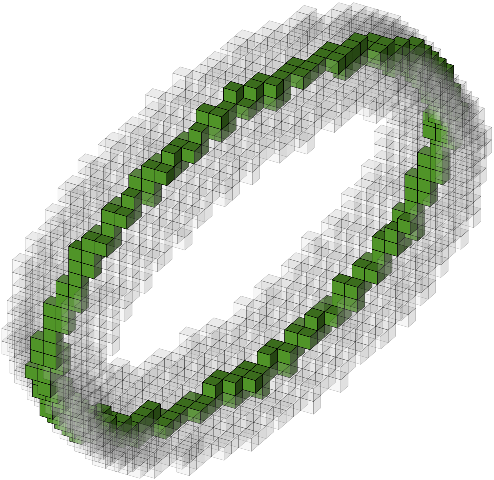




SN $(2, 1)$

# DSGP Topological Classes





SS $(2, 2)$

# DSGP Topological Classes



## Theorem (Class Bounds)

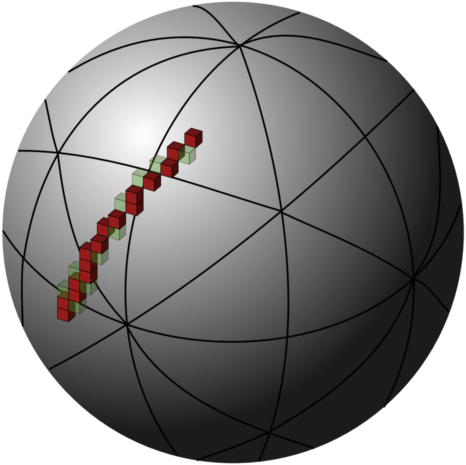
*The respective upper bounds of the isothetic distance of the DSGP $\boldsymbol{\pi}_{m,n}^{(l)}(s,t)$ from the real sphere $S$ and of that from the real plane $\Pi_r^{\mathbb{R}}(s,t)$ for classes* NS, SN*, and* SS *are as follows.*

$$\max_{p \in \boldsymbol{\pi}_{m,n}^{(l)}(s,t)} \{d_\perp(p,S)\} \begin{cases} < \frac{1}{2} & \textit{if } l \in \{0,1\}, m=1, n=2 \\ \leqslant 2 & \textit{if } l \in \{0,1\}, m=2, n=1 \\ \leqslant 2 & \textit{if } l \in \{0,1,2\}, m=2, n=2 \end{cases}$$

$$\max_{p \in \boldsymbol{\pi}_{m,n}^{(l)}(s,t)} \{d_\perp(p,\Pi_r^{\mathbb{R}}(s,t))\} \begin{cases} \leqslant \frac{3}{2} & \textit{if } l \in \{0,1\}, m=1, n=2 \\ \leqslant \frac{1}{2} & \textit{if } l \in \{0,1\}, m=2, n=1 \\ \leqslant \frac{3}{2} & \textit{if } l \in \{0,1,2\}, m=2, n=2 \end{cases}$$

# DSGP Topological Classes

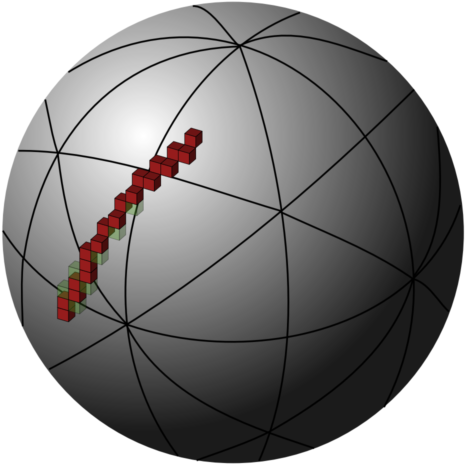




NS $(l = 0) : 16$

# DSGP Topological Classes

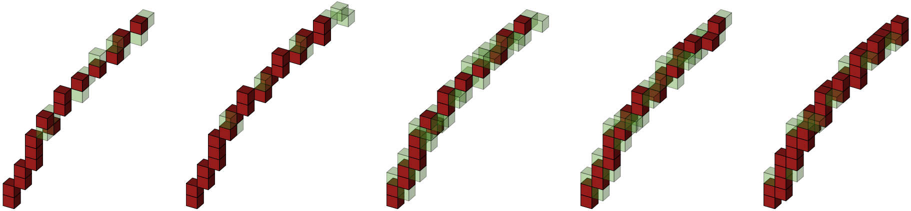




NS $(l = 1) : 19$

# DSGP Topological Classes





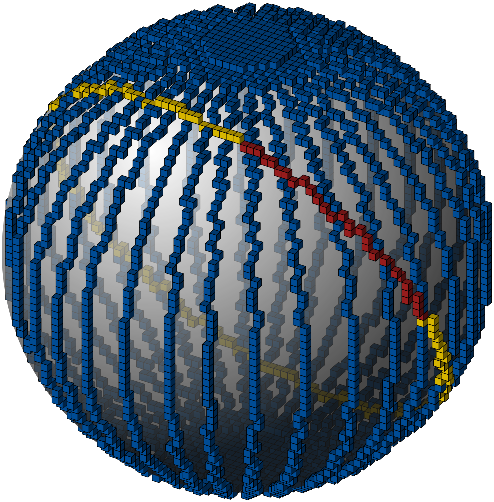
SN $(l = 1) : 16$ | SN $(l = 1) : 19$ | SS $(l = 0) : 16$ | SS $(l = 1) : 18$ | SS $(l = 2) : 30$

**Different classes of DSGP (red voxels)**

$(r = 17, s = (-6, -1, 16), t = (2, 14, 10))$

# DSGP Topological Classes

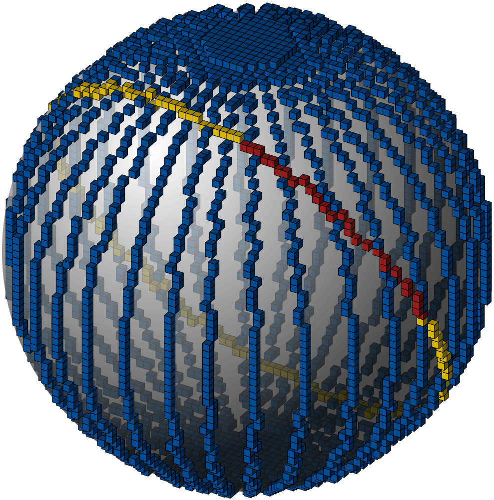




$\mathsf{NS}(l = 1),\ r = 30$

# DSGP Topological Classes



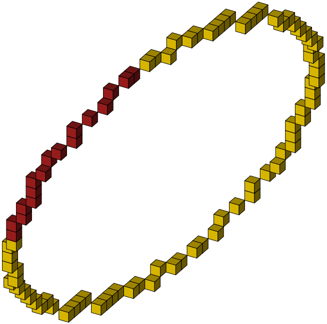


NS($l = 0$), $r = 30$

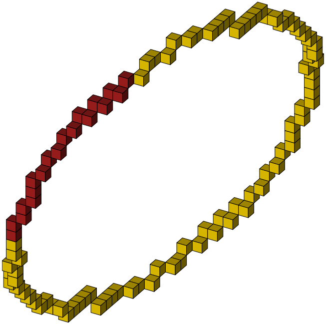
# Open Problems



- *Discrete 3D circles*—maximum symmetry + minimum length-and-deviation.



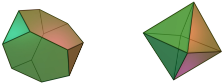
NS $(0, 1, 2)$ : 90 vox

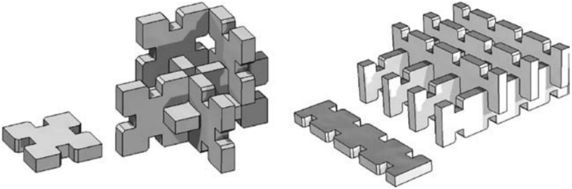


NS $(1, 1, 2)$ : 102 vox

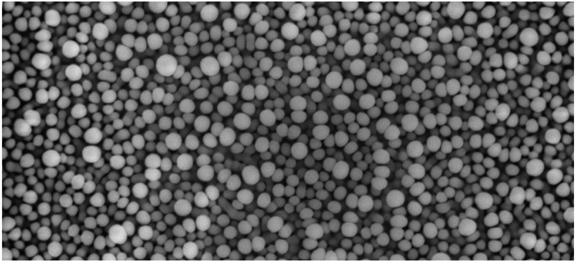
# Open Problems (2)

- *Voxel strengthening*—for improved 3D printing, by reshaping voxel as truncated tetrahedron, octahedron, sphere, or even Great Invention Kit (GIKs) [6].





# Open Problems (3)



A microscopic view of rounded crystals
produced by the scientists for 3d-printing

# Open Problems



- *iso-contours, geodesic distance query*—as in 3D real space [10, 11].
- *Rational specification*—characterization and algorithm.

# Further Reading (1)


- E. Andres.
  Discrete circles, rings and spheres.
  *Computers & Graphics*, 18(5):695–706, 1994.
- E. Andres and M. Jacob.
  The discrete analytical hyperspheres.
  *IEEE Trans. Visualization and Computer Graphics*, 3(1):75–86, 1997.
- R. Biswas and P. Bhowmick.
  Layer the sphere.
  *The Visual Computer*, 31:787–797, 2015.
- R. Biswas and P. Bhowmick.
  On different topological classes of spherical geodesic paths and circles in $\mathbb{Z}^3$.
  *Theoretical Computer Science*, 605:146–163, 2015.

# Further Reading




R. Biswas and P. Bhowmick.
From prima quadraginta octant to lattice sphere through primitive integer operations.
*Theoretical Computer Science*, 2015 (in press, doi: `http://dx.doi.org/10.1016/j.tcs.2015.11.018`).

J. Hiller and H. Lipson.
Design and analysis of digital materials for physical 3D voxel printing.
*Rapid Prototyping Journal*, 15(2):137–149, 2009.

C. Montani and R. Scopigno.
Graphics gems (Chapter: *Spheres-to-voxels conversion*), A. S. Glassner (Ed.).
pages 327–334. Academic Press Professional, Inc., San Diego, CA, USA, 1990.

B. Roget and J. Sitaraman.
Wall distance search algorithm using voxelized marching spheres.
*Journal of Computational Physics*, 241:76–94, 2013.

# Further Reading (3)


- J.-L. Toutant, E. Andres, and T. Roussillon.
  Digital circles, spheres and hyperspheres: From morphological models to analytical characterizations and topological properties.
  *Discrete Applied Mathematics*, 161(16-17):2662–2677, 2013.

- S.-Q. Xin, X. Ying, and Y. He.
  Constant-time all-pairs geodesic distance query on triangle meshes.
  In *ACM SIGGRAPH Symposium on Interactive 3D Graphics and Games*, pages 31–38, 2012.

- X. Ying, X. Wang, and Y. He.
  Saddle Vertex Graph (SVG): A novel solution to the discrete geodesic problem.
  *ACM Trans. Graphics*, 32(6):170:1–170:12, 2013.

*Thank You*