From circle to sphere and to related problems in the digital space

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Circle Construction

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From circle to sphere and to related problems in the digital space
This is $\mathbb{Z}^2$—an infinite set of 2D integer points
Construction by Digitization

This is a real circle
(integer center, radius 14)
Construction by Digitization

This is the *digital circle* (integer center, radius 14)
Construction by Digitization

distance \( < \frac{1}{2} \)
The challenge

Given \( r \) and \( c \) as integers, use only integer arithmetic to compute the digital circle. (w.l.o.g., \( c = (0, 0) \))
Construction by Digitization

As displayed on computer screen/book

$r = 14$
Construction by Digitization

As displayed on computer screen/book

\[ r = 14 \]
## Construction by Digitization

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Inventors</th>
<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incremental</td>
<td>Bresenham</td>
<td>1977</td>
</tr>
<tr>
<td>Optimized midpoint</td>
<td>Foley <em>et al.</em></td>
<td>1993</td>
</tr>
<tr>
<td>Short run</td>
<td>Hsu <em>et al.</em></td>
<td>1993</td>
</tr>
<tr>
<td>Hybrid run slice</td>
<td>Yao &amp; Rokne</td>
<td>1995</td>
</tr>
<tr>
<td><em><em>Number theory</em>^a</em>*</td>
<td>Bhowmick &amp; Bhattacharya</td>
<td>2008</td>
</tr>
</tbody>
</table>

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*^a P. Bhowmick and B. B. Bhattacharya,
Number-theoretic interpretation and construction of a digital circle,
*Discrete Applied Mathematics*, **156**: 2381–2399, **2008**.
#permutations of \((i, j)\) including sign \(= 2 \times 2^2 = 8\).
A simple question: What’s the pattern here? (Disregard the 1st line)

0, 13
14, 39
40, 63
64, 85
86, 105

...
A simple question: What’s the pattern here? (Disregard the 1st line)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>length</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 13</td>
<td></td>
<td>26</td>
</tr>
<tr>
<td>14, 39</td>
<td></td>
<td>24</td>
</tr>
<tr>
<td>40, 63</td>
<td></td>
<td>22</td>
</tr>
<tr>
<td>64, 85</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>86, 105</td>
<td></td>
<td>...</td>
</tr>
</tbody>
</table>
A simple question: What’s the pattern here? (Disregard the 1st line)

\[
\begin{align*}
0, 13 & \rightarrow 26 \quad \rightarrow -2 \\
14, 39 & \rightarrow 24 \quad \rightarrow -2 \\
40, 63 & \rightarrow 22 \quad \rightarrow -2 \\
64, 85 & \rightarrow 20 \quad \rightarrow -2 \\
86, 105 & \rightarrow 20 \\
\ldots
\end{align*}
\]
A simple question: What’s the pattern here? (Disregard the 1st line)

0, 13
14, 39 → 26
40, 63 → 24
64, 85 → 22
86, 105 → 20

...
### Number-theoretic Properties

A simple question: What’s the pattern here? (Disregard the 1st line)

<table>
<thead>
<tr>
<th>0, 13</th>
<th>14, 39 → 26</th>
</tr>
</thead>
<tbody>
<tr>
<td>40, 63 → 24</td>
<td></td>
</tr>
<tr>
<td>64, 85 → 22</td>
<td></td>
</tr>
<tr>
<td>86, 105 → 20</td>
<td></td>
</tr>
</tbody>
</table>

···

\[ r = 14 \]

This is how we get the digital circle for \( r = 14 \)

---

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From circle to sphere and to related problems in the digital space
Number-theoretic Properties

A simple question: What’s the pattern here? (Disregard the 1st line)

\[
\begin{align*}
I_0 &= [0, r-1] \\
I_1 &= [r, 3r-3] \\
I_2 &= [3r-2, 5r-7] \\
&\vdots \\
I_k &= [u_k, u_k + l_k - 1] \\
&\vdots
\end{align*}
\]

\[
\begin{align*}
0, 13 &\to 26 \\
14, 39 &\to 24 \\
40, 63 &\to 22 \\
64, 85 &\to 20 \\
86, 105 &\to \ldots
\end{align*}
\]
Number-theoretic Properties

A simple question: What's the pattern here? (Disregard the 1st line)

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\begin{align*}
I_0 &= [0, r-1] \\
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&\vdots \\
I_k &= [u_k, u_k + l_k - 1] \\
&\vdots
\end{align*}
\]

Recurrence for \( r \):

\[
\begin{align*}
u_k &= u_{k-1} + l_{k-1} \quad \text{if } k \geq 1 \\
0 &\quad \text{if } k = 0
\end{align*}
\]

\[
\begin{align*}
l_k &= l_{k-1} - 2 \quad \text{if } k \geq 2 \\
2r - 2 &\quad \text{if } k = 1 \\
r &\quad \text{if } k = 0
\end{align*}
\]
Number-theoretic Properties

A simple question: What’s the pattern here? (Disregard the 1st line)

\[
\begin{align*}
0, & \ 13 \\
14, & \ 39 \quad \rightarrow \quad 26 \\
40, & \ 63 \quad \rightarrow \quad 24 \\
64, & \ 85 \quad \rightarrow \quad 22 \\
86, & \ 105 \quad \rightarrow \quad 20
\end{align*}
\]

\[
\begin{align*}
I_0 &= [0, r - 1] \\
I_1 &= [r, 3r - 3] \\
I_2 &= [3r - 2, 5r - 7] \\
& \quad \ldots \\
I_k &= [u_k, u_k + l_k - 1] \\
& \quad \ldots
\end{align*}
\]

Recurrence for \( r \)

\[
\begin{align*}
u_k &= \begin{cases} 
  u_{k-1} + l_{k-1} & \text{if } k \geq 1 \\
  0 & \text{if } k = 0
\end{cases} \\
l_k &= \begin{cases} 
  l_{k-1} - 2 & \text{if } k \geq 2 \\
  2r - 2 & \text{if } k = 1 \\
  r & \text{if } k = 0
\end{cases}
\end{align*}
\]

**Theorem** \( I_k \) contains the \( x^2 \) of the points at \( k \)th run.
Number-theoretic Properties

\( r = 14 : \)

- \( k = 0 : [0, 13] \Rightarrow 4 \text{ squares} \)
- \( k = 1 : [14, 39] \Rightarrow 3 \text{ squares} \)
- \( k = 2 : [40, 63] \Rightarrow 1 \text{ square} \)
- \( k = 3 : [64, 85] \Rightarrow 2 \text{ squares} \)
- \( k = 4 : [86, 105] \Rightarrow 1 \text{ square} \)
Algorithm **DCS** (int \( r \))

1. \( \text{int } i \leftarrow 0, j \leftarrow r, s \leftarrow 0, w \leftarrow r - 1, l \leftarrow 2r - 2 \)
2. **while** \( j \geq i \)
3. **do**
4. select \((i, j)\)
5. \( s \leftarrow s + 2i + 1 \)
6. \( i \leftarrow i + 1 \)
7. **while** \( s \leq w \)
8. \( w \leftarrow w + l \)
9. \( l \leftarrow l - 2 \)
10. \( j \leftarrow j - 1 \)

---

\( k = 0 \):
\[ [0, r - 1] = [0, 13] \Rightarrow 4 \]
\( k = 1 \):
\[ [r, 3r - 3] = [14, 39] \Rightarrow 3 \]
\( k = 2 \):
\[ [3r - 2, 5r - 7] = [40, 63] \Rightarrow 1 \]
\( k = 3 \):
\[ [5r - 6, 7r - 13] = [64, 85] \Rightarrow 2 \]
\( k = 4 \):
\[ [7r - 12, 9r - 21] = [86, 105] \Rightarrow 1 \]

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More number-theoretic properties

A simple observation

Let $u, v, w$ be three positive integers in increasing order such that $w - v = v - 1 - u$.

Let

$s[u, v - 1] = \#\text{squares in } [u, v - 1]$,

$s[v, w] = \#\text{squares in } [v, w]$.

Then can $s[v, w] > s[u, v - 1]$? If so, by how much?
More number-theoretic properties

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More number-theoretic properties

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From circle to sphere and to related problems in the digital space
More number-theoretic properties

Lemma

For $u < v < w$ and $w - v = v - 1 - u$, $s[v, w] \leq s[u, v - 1] + 1$.

Hence, a useful result:
For $u < v < w$ and $w - v = v - u - 3$, $s[v, w] \leq s[u, v - 1] + 1$.

And so the theorem follows in next slide!
More number-theoretic properties

**Theorem (Upper bound of run length ($\lambda$))**

$$\lambda(j - 1) \leq \lambda(j) + 1.$$ 

$r = 41$
More number-theoretic properties

Lemma

For $u < v < w$ and $w - v = v - 1 - u$, $s[v, w] \geq \left\lfloor \frac{s[u,v-1]-1}{2} \right\rfloor$.

And so the theorem follows in next slide!
More number-theoretic properties

Theorem (Lower bound of run length ($\lambda$))

$$\lambda(j - 1) \geq \left\lfloor \frac{\lambda(j) - 1}{2} \right\rfloor - 1$$

$r = 41$
More number-theoretic properties

Constructive bounds

\[
\left\lfloor \frac{\lambda(j) - 1}{2} \right\rfloor - 1 \leq \lambda(j - 1) \leq \lambda(j) + 1
\]

\(r = 41\)
Algorithm **DCR**

Demonstration of **DCR** for $r = 106$. 
**Algorithm DCR: Square search**

Algorithm DCR (int r) {
1. int i = 0, j = r, w = r - 1, m;
2. int s = 0, t = r, l = w << 1;
3. while (j ≥ i) {
   4. while (s < t) {
      5. m = s + t;
      6. m = m >> 1;
      7. if (w ≤ square[m])
         8. t = m;
      9. else
         10. s = m + 1; }
   11. if (w < square[s])
          s --;
   12. s ++;
   13. include_run (i, s - i, j);
   14. t = s + s - i + 1;
   15. i = s;
   16. w = w + l;
   17. l = l - 2;
   18. j --; }
}
Hybrid algorithm DCH

Algorithm DCH (int r, int p) {
1. int i = 0, j = r, w = r − 1, m;
2. int s = 0, t = r, l = w << 1;
3. while (j ≥ i) {
4.   while (s < t) {
5.     m = s + t;
6.     m = m >> 1;
7.     if (w ≤ square[m])
8.       t = m;
9.     else
10.         s = m + 1;
11.     if (w < square[s])
12.       s − −;
13.     s + +;
14.     include_run (i, s − i, j);
15.     if (s − i < p)
16.       break;
17.     t = s + s − i + 1;
18.     i = s;
19.     w = w + l;
20.     l = l − 2;
21.     j − −; }
22.   i = s − 1;
23.   s = square[s];
24.   w = w + l;
25.   l = l − 2;
26.   j − −;
27.   while (j ≥ i) {
28.     do {sym_8 (i, j);
29.     s = s + i;
30.     i + +;
31.     s = s + i; } while (s ≤ w);
32.   w = w + l;
33.   l = l − 2;
34.   j − −; }}
Test Results...

DCB
Test Results...

DCR

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From circle to sphere and to related problems in the digital space
Digital Circularity
Does there exist a real circle (integer radius & center) such that each point of the given sequence lies within a distance of \( \frac{1}{2} \) from it?
47 is far from true.
Seems, it will be much larger!
But how large? And how to get it?
170 is *the* solution!
How to get it very fast, using simple arithmetic (no trigonometry etc.)?
Conflicting Radii

\[ \lambda_0 = 6 \]

\[ r \in [26, 36] \]
Conflicting Radii

Lemma

\( \lambda_0 \) is the length of top run of a digital circle \( C^\mathbb{Z}(o, r) \) iff \( r \in R_0 := [(\lambda_0 - 1)^2 + 1, \lambda_0^2] \).
Conflicting Radii

\[ r \in [26, 27] \]
Conflicting Radii

\[ \lambda_0 = 6 \quad \lambda_1 = 4 \]

\[ r = 34 \]

\[ r = 28 \]

\[ r \in [28, 34] \]
Conflicting Radii

\[ r = 36 \]
\[ \lambda_0 = 6 \]
\[ \lambda_1 = 5 \]

\[ r \in [35, 36] \]
Radii Nesting

\[ \lambda_0 = 6 \]
\[ \lambda_1 = 3 \]
\[ r = 26 \]
\[ r = 27 \]
\[ r = 28 \]

\[ \lambda_0 = 6 \]
\[ \lambda_1 = 4 \]
\[ r = 28 \]
\[ r = 34 \]

\[ \lambda_0 = 6 \]
\[ \lambda_1 = 5 \]
\[ r = 35 \]
\[ r = 36 \]

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Lemma

$\lambda_0$ and $\lambda_1$ are the lengths of top two runs of $C^\mathbb{Z}(o, r)$ iff $r \in R_0 \cap R_1$, where, $R_1 = \left[\left\lfloor \frac{(\Lambda_1 - 1)^2 + 3}{3} \right\rfloor, \left\lfloor \frac{\Lambda_1^2 + 2}{3} \right\rfloor\right]$, $\Lambda_1 = \lambda_0 + \lambda_1$.

(If $R_0 \cap R_1 = \emptyset$, then there exists no digital circle ... $\lambda_0$ and $\lambda_1$.)
Theorem (Radii interval)

\[ \langle \lambda_0, \ldots, \lambda_n \rangle \text{ is the sequence of top } n + 1 \text{ run-lengths of } C^\mathbb{Z}(o, r) \text{ iff} \]

\[ r \in \bigcap_{k=0}^{n} R_k \]

where,

\[ R_k = \left[ \left\lfloor \frac{1}{2k + 1} \left( (\Lambda_k - 1)^2 + k(k + 1) + 1 \right) \right\rfloor, \left\lceil \frac{1}{2k + 1} \left( \Lambda_k^2 + k(k + 1) \right) \right\rceil \right] \]

and

\[ \Lambda_k = \sum_{j=0}^{k} \lambda_j. \]

(If \( \bigcap_{k=0}^{n} R_k = \emptyset \), then there exists no digital circle whose top \( n + 1 \) runs have length \( \langle \lambda_0, \lambda_1, \ldots, \lambda_n \rangle \).)
Algorithm DCT

1. \( \Lambda \leftarrow S[0] \)
2. \([r', r''] \leftarrow [(\Lambda - 1)^2 + 1, \Lambda^2] \)
3. for \( k \leftarrow 1 \) to \( n - 1 \)
4. \( \Lambda \leftarrow \Lambda + S[k] \)
5. \( s' \leftarrow \left\lfloor \frac{((\Lambda - 1)^2 + k(k + 1) + 1)/(2k + 1)}{2k + 1} \right\rfloor \)
6. \( s'' \leftarrow \left\lfloor \frac{(\Lambda^2 + k(k + 1))/(2k + 1)}{2k + 1} \right\rfloor \)
7. if \( s'' < r' \) or \( s' > r'' \)
8. print “\( S \) is circular up to \((k - 1)\)th run for \([r', r'']\).”
9. return
10. else
11. \([r', r''] \leftarrow [\max(r', s'), \min(r'', s'')] \)
12. print “\( S \) is circular in entirety for \([r', r'']\).”
Conflicting Radii: Resolved how fast?
Conflicting Radii: Resolved how fast?

Conflicting radii starting from $k = 0$
Conflicting Radii: Resolved how fast?

Resolving the conflicting radii $r'$ with increasing $k$

$k = 1$
Conflicting Radii: Resolved how fast?

Resolving the conflicting radii $r'$ with increasing $k$

$k = 2$
Conflicting Radii: Resolved how fast?

Resolving the conflicting radii $r'$ with increasing $k$

$k = 3$
Resolving the conflicting radii $r'$ with increasing $k$

$k = 4$
Lemma

If a digital circle of radius $r$ contains a given run of length $\lambda$, then there exist two positive integers $a$ and $k$ such that $r \geq \lceil \max \left( f_{1,\lambda}(a, k), f_{2,\lambda}(a, k) \right) \rceil$, where

$$f_{1,\lambda}(a, k) = \frac{(a - 1)^2 + k(k - 1) + 1}{2k - 1}$$

and

$$f_{2,\lambda}(a, k) = \frac{(a + \lambda - 1)^2 + k(k + 1) + 1}{2k + 1}.$$
General Case & DCG

Lemma

If a digital circle of radius $r$ contains a given run of length $\lambda$, then there exist two positive integers $a$ and $k$ such that

$r \leq \lfloor \min (f_3,\lambda(a,k), f_4,\lambda(a,k)) \rfloor$, where

\[ f_{3,\lambda}(a,k) = \frac{a^2 + k(k - 1)}{2k - 1} \]

and

\[ f_{4,\lambda}(a,k) = \frac{(a + \lambda)^2 + k(k + 1)}{2k + 1}. \]
Theorem

An arbitrary run of given length $\lambda$ belongs to only those digital circles whose radii are in the range

$$\mathcal{R}_{ak} = \left\{ r \mid r \geq \max_{a,k \in \mathbb{Z}^+} (f_1,\lambda(a,k),f_2,\lambda(a,k)) \right\} \cap \left\{ r \mid r \leq \min_{a,k \in \mathbb{Z}^+} (f_3,\lambda(a,k),f_4,\lambda(a,k)) \right\}.$$
From circle to sphere and to related problems in the digital space

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Points of intersection (in $\mathbb{R}^2$) among the parabolas
\( \{f_i, \lambda \mid i = 1, 2, 3, 4\} \) defining $\mathcal{R}_{ak}$.

\[
(k = 2k - 1, \bar{k} = 2k + 1, \hat{k} = k(k - 1), \hat{\lambda} = k(k + 1), \lambda = \lambda - 1)
\]

<table>
<thead>
<tr>
<th>Parabolas</th>
<th>Point</th>
<th>Abscissa of the point</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1, \lambda$</td>
<td>$f_2, \lambda$</td>
<td>$\alpha_{12}$</td>
</tr>
<tr>
<td>$f_2, \lambda$</td>
<td>$f_3, \lambda$</td>
<td>$\alpha_{23}$</td>
</tr>
<tr>
<td>$f_3, \lambda$</td>
<td>$f_4, \lambda$</td>
<td>$\alpha_{34}$</td>
</tr>
<tr>
<td>$f_4, \lambda$</td>
<td>$f_1, \lambda$</td>
<td>$\alpha_{41}$</td>
</tr>
</tbody>
</table>
**Specifications of the parabolas** \( \{ f_{i, \lambda} \mid i = 1, 2, 3, 4 \} \).

<table>
<thead>
<tr>
<th>Parabola</th>
<th>Axis</th>
<th>Directrix</th>
<th>Length of Latus Rectum</th>
<th>Vertex</th>
<th>Focus</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{1, \lambda} )</td>
<td>( x = 1 )</td>
<td>( k y = 3/4 )</td>
<td>( k )</td>
<td>( 1, (\hat{k} + 1)/k )</td>
<td>( 1, (8\hat{k} + 5)/(4k) )</td>
</tr>
<tr>
<td>( f_{2, \lambda} )</td>
<td>( x = -\lambda )</td>
<td>( k y = 3/4 )</td>
<td>( \bar{k} )</td>
<td>( -\lambda, (\hat{k} + 1)/\bar{k} )</td>
<td>( -\lambda, (8\hat{k} + 5)/(4\bar{k}) )</td>
</tr>
<tr>
<td>( f_{3, \lambda} )</td>
<td>( x = 0 )</td>
<td>( k y = -1/4 )</td>
<td>( k )</td>
<td>( 0, (\hat{k})/k )</td>
<td>( 0, (8\hat{k} + 1)/(4k) )</td>
</tr>
<tr>
<td>( f_{4, \lambda} )</td>
<td>( x = -\lambda )</td>
<td>( \bar{k} y = -1/4 )</td>
<td>( \bar{k} )</td>
<td>( -\lambda, \bar{k}/\hat{k} )</td>
<td>( -\lambda, (8\hat{k} + 1)/(4\bar{k}) )</td>
</tr>
</tbody>
</table>
General Case & DCG

**Specifications of the parabolas** \( \{f_{i,\lambda} \mid i = 1, 2, 3, 4\} \).

**Points of intersection (in \( \mathbb{R}^2 \)) among the parabolas** \( \{f_{i,\lambda} : i = 1, 2, 3, 4\} \) defining \( R_{ak} \).

To obtain the value of \( \{\alpha_{ij} \mid j = (i \text{ mod } 4) + 1, i = 1, 2, 3, 4\} \), we have solved the following quadratic equations in \( a \). Out of the two values of \( a \) obtained, say \( a = C \pm \sqrt{D} \), we define \( \alpha \) as \( C + \sqrt{D} \).

\[
\alpha_{23} : \quad \frac{(a+\lambda-1)^2 + k(k+1) + 1}{2k+1} = \frac{a^2 + k(k-1)}{2k-1}
\]

or, \( (2k - 1)(a^2 + 2(\lambda - 1)a + (\lambda - 1)^2 + k(k + 1) + 1) = (2k + 1)(a^2 + k(k - 1)) \)

or, \( 2a^2 - 2(2k - 1)(\lambda - 1)a - (2k - 1)(\lambda - 1)^2 - 2k^2 - 2k + 1 = 0 \)

or, \( a = \frac{1}{2} \left( (2k - 1)(\lambda - 1) \pm \sqrt{(2k - 1)^2(\lambda - 1)^2 + 2((2k - 1)(\lambda - 1)^2 + 2k^2 + 2k - 1)} \right) \)

or, \( \alpha_{23} = \frac{1}{2} \left( (2k - 1)(\lambda - 1) + \sqrt{(2k - 1)^2(\lambda - 1)^2 + 2((2k - 1)(\lambda - 1)^2 + 2k^2 + 2k - 1)} \right) \).

\[
\alpha_{12} : \quad \frac{(a-1)^2 + k(k-1) + 1}{2k-1} = \frac{(a+\lambda-1)^2 + k(k+1) + 1}{2k+1}
\]

or, \( (2k + 1)((a - 1)^2 + k(k - 1) + 1) = (2k - 1)((a + \lambda - 1)^2 + k(k + 1) + 1) \)

or, \( 2a^2 - 2((2k - 1)\lambda + 2)a - (2k - 1)(\lambda - 1)^2 - 2k^2 + 2k + 3 = 0 \)

or, \( a = \frac{1}{2} \left( (2k - 1)\lambda + 2 \pm \sqrt{((2k - 1)\lambda + 2)^2 + 2((2k - 1)(\lambda - 1)^2 + 2k^2 - 2k - 3)} \right) \)

or, \( \alpha_{12} = \frac{1}{2} \left( (2k - 1)\lambda + 2 + \sqrt{((2k - 1)\lambda + 2)^2 + 2((2k - 1)(\lambda - 1)^2 + 2k^2 - 2k - 3)} \right) \).
General Case & DCG

\(\alpha_{41}:\) \(\frac{(a+\lambda)^2+k(k+1)}{2k+1} = \frac{(a-1)^2+k(k-1)+1}{2k-1}\)

or, \((2k - 1)((a + \lambda)^2 + k(k + 1)) = (2k + 1)((a - 1)^2 + k(k - 1) + 1)\)

or, \(2a^2 - 2(2k(1 + \lambda) - \lambda + 1)a - (2k - 1)\lambda^2 - 2k^2 + 4k + 2 = 0\)

or, \(a = \frac{1}{2} \left( (2k - 1)\lambda + 2k + 1 \pm \sqrt{(2k - 1)\lambda + 2k + 1)^2 + 2((2k - 1)\lambda^2 + 2k^2 - 4k - 2) \right) \)

or, \(\alpha_{41} = \frac{1}{2} \left( (2k - 1)\lambda + 2k + 1 + \sqrt{(2k - 1)\lambda + 2k + 1)^2 + 2((2k - 1)\lambda^2 + 2k^2 - 4k - 2) \right) \).

\(\alpha_{34}:\) \(\frac{a^2+k(k-1)}{2k-1} = \frac{(a+\lambda)^2+k(k+1)}{2k+1}\)

or, \((2k + 1)(a^2 + k(k - 1)) = (2k - 1)((a + \lambda)^2 + k(k + 1))\)

or, \(2a^2 - (2k - 1)\lambda - (2k - 1)\lambda^2 - 2k^2 = 0\)

or, \(a = \frac{1}{2} \left( (2k - 1)\lambda \pm \sqrt{(2k - 1)^2\lambda^2 + 2((2k - 1)\lambda^2 + 2k^2) \right) \)

or, \(\alpha_{34} = \frac{1}{2} \left( (2k - 1)\lambda + \sqrt{(2k - 1)^2\lambda^2 + 2((2k - 1)\lambda^2 + 2k^2) \right) \).
Algorithm DCG

1. \(n_{\text{max}} \leftarrow 0\)
2. \(\text{for } k' \leftarrow k_{\min} \text{ to } k_{\max}\)
3. \(\Lambda \leftarrow S[0], i \leftarrow 0\)
4. \(\text{FIND-PARAMS}(A, \Lambda, k')\)
5. \(\text{while } i < m \text{ and } n_{\text{max}} < n \triangleright \text{for all } a's \text{ of first run}\)
6. \([s', s''] \leftarrow [r', r''] \leftarrow [A[i][1], A[i][2]]\)
7. \(\Lambda \leftarrow A[i][0] + S[0], j \leftarrow 1\)
8. \(\text{while } j < n \text{ and } s'' \geq r' \text{ and } s' \leq r'' \triangleright \text{verifying other } n - 1 \text{ runs}\)
9. \(\Lambda \leftarrow \Lambda + S[j], k \leftarrow k' + j\)
10. \(s' \leftarrow \left\lceil \frac{(\Lambda-1)^2+k(k+1)+1}{2k+1} \right\rceil, s'' \leftarrow \left\lfloor \frac{\Lambda^2+k(k+1)}{2k+1} \right\rfloor\)
11. \(\text{if } s'' \geq r' \text{ and } s' \leq r''\)
12. \([r', r''] \leftarrow [\max(r', s'), \min(r'', s'')]\)
13. \(\text{if } n_{\text{max}} < j\)
14. \(n_{\text{max}} \leftarrow j, k_{\text{off}} \leftarrow k', [r_{\min}, r_{\max}] \leftarrow [r', r'']\)
15. \(\text{print } \text{"S is circular for } n_{\text{max}} \text{ runs; starting run } = k_{\text{off}}; r \in [r_{\min}, r_{\max}]\text{."} \)
Procedure FIND-PARAMS

1. Compute \( \{\alpha_{uv} \mid 1 \leq u \leq 4 \land v = (u + 1) \mod 4\} \) (from Tables)
2. \( i \leftarrow 0 \)
3. for \( a \leftarrow \lceil \alpha_{23} \rceil \) to \( \lfloor \alpha_{41} \rfloor \)
4. \( A[i][0] \leftarrow a \) computing \( r' \)
5. if \( a < \alpha_{12} \)
6. \( A[i][1] \leftarrow \lceil f_2, \lambda (a, k) \rceil \)
7. else
8. \( A[i][1] \leftarrow \lfloor f_1, \lambda (a, k) \rfloor \) computing \( r'' \)
9. if \( a < \alpha_{34} \)
10. \( A[i][2] \leftarrow \lfloor f_3, \lambda (a, k) \rfloor \)
11. else
12. \( A[i][2] \leftarrow \lceil f_4, \lambda (a, k) \rceil \)
13. \( i \leftarrow i + 1 \)
14. \( m \leftarrow i \)
Algorithm \textit{DCG}

\begin{align*}
A[2][1] &= 101 \\
A[1][2] &= 97 \\
A[1][1] &= 87 \\
A[0][2] &= 81 \\
A[0][1] &= 76
\end{align*}

\[ A[0][0] = \lfloor \alpha_{23} \rfloor \lfloor \alpha_{34} \rfloor \lfloor \alpha_{12} \rfloor = \lfloor \alpha_{41} \rfloor = A[2][0] \]

\textbf{FIND-PARMS} on a run-length 7:

Solution space \( \mathcal{R}_{ak} \) of the radius intervals \( \{ [r'_j, r''_j] \mid j = 0, 1, 2 \} \) corresponding to \( m = 3 \) square numbers lying in \( \left[ \lfloor \alpha_{23} \rfloor^2, \lfloor \alpha_{41} \rfloor^2 \right] = [9^2, 11^2] \).
Arc Segmentation

Partha Bhowmick

From circle to sphere and to related problems in the digital space
Arc Segmentation

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Arc Segmentation

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## Arc Segmentation

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Inventors</th>
<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Voronoi diagram</td>
<td>Coeurjolly <em>et al.</em></td>
<td>2004</td>
</tr>
<tr>
<td><em>Chord &amp; Sagitta</em></td>
<td>Bera, Bhowmick &amp; Bhattacharya</td>
<td>2010</td>
</tr>
<tr>
<td><em>Discrete Curvature</em>&lt;sup&gt;a&lt;/sup&gt;</td>
<td>Pal, Dutta &amp; Bhowmick</td>
<td>2012</td>
</tr>
<tr>
<td><em>Number Theory</em>&lt;sup&gt;b&lt;/sup&gt;</td>
<td>Pal &amp; Bhowmick</td>
<td>2012</td>
</tr>
<tr>
<td><em>Number Theory &amp; Graph Theory</em>&lt;sup&gt;c&lt;/sup&gt;</td>
<td>Bhowmick &amp; Pal</td>
<td>2014</td>
</tr>
</tbody>
</table>

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From circle to sphere and to related problems in the digital space
From circle to sphere and to related problems in the digital space
From circle to sphere and to related problems in the digital space
From circle to sphere and to related problems in the digital space
Discretization of Sphere
Lattice, cells, voxels, adjacency

From circle to sphere and to related problems in the digital space
Lattice, cells, voxels, adjacency

- **lattice point**
  - (integer coordinates)

- **lattice line**
  - 2-cell $\equiv$ face
  - 1-cell $\equiv$ edge
  - 0-cell $\equiv$ vertex

- **3-cell $\equiv$ voxel**
  - (8 vertices, 12 edges, 6 faces)
Lattice, cells, voxels, adjacency

From circle to sphere and to related problems in the digital space
Discretization Models (*general surface*)

naïve
Discretization Models\textit{ (general surface)}

\textit{naive}
Discretization Models \textit{(general surface)}

\begin{itemize}
  \item \textit{naive}
  \item no 2-path
\end{itemize}
Discretization Models (general surface)

- naive
- no 2-path
- 1-path

From circle to sphere and to related problems in the digital space
Discretization Models *(general surface)*

- **naive**
- **no 2-path**
- **1-path**

- **standard**
Discretization Models (*general* *surface*)

- **naive**
- **no 2-path**
- **1-path**

From circle to sphere and to related problems in the digital space
Discretization Models *(general surface)*

- **naive**
- no 2-path
- 1-path

- **standard**
- no 0-path
Discretization Models \textit{(general surface)}

\textit{Naive} = 2-minimal. \textit{Standard} = 0-minimal.
**Problem Statement:** Given integer radius and integer center\(^1\), construct the *naive sphere* whose *every voxel* is *non-redundant* and *lies as much close as possible to the real sphere.*

\(^1\)w.l.o.g., center = (0, 0, 0)
Non-redundant
To formalize “as much close as possible to the real sphere”, we define 
\[ d_\perp(p, S) = \min\{d_x(p, S), d_y(p, S), d_z(p, S)\}. \]
**Discretization Models** *(plane & sphere)*

**Real plane** $\Pi(a, b, c, \mu) : ax + by + cz = \mu$.

**Digital plane**

$\Pi^\mathbb{Z}(a, b, c, \mu, \omega) = \{(i, j, k) \in \mathbb{Z}^3 : \mu - \frac{\omega}{2} \leq ai + bj + ck < \mu + \frac{\omega}{2}\}$, which is of *thickness* $\omega$ and centered on $\Pi$.

**Example:** $6x + 13y + 27z = 0$

\[\omega = 15 \quad \omega = 27 \quad \omega = 46\]
## Discretization Models *(plane & sphere)*

<table>
<thead>
<tr>
<th>under-digitized</th>
<th>naive</th>
<th>standard</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega &lt; \max(</td>
<td>a</td>
<td>,</td>
</tr>
</tbody>
</table>

Example: \( 6x + 13y + 27z = 0 \)

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( \omega )</th>
<th>( \omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>27</td>
<td>46</td>
</tr>
</tbody>
</table>
Discretization Models \((plane & sphere)\)

**Lemma**

For a point \(p = (i, j, k)\) and a real plane \(\Pi : ax + by + cz = 0\),
\[
d_\perp(p, \Pi) = \frac{|ai + bj + ck|}{\max(|a|, |b|, |c|)}.
\]

**Theorem (Point-to-Plane Distance\(^a\))**

\[
d_\perp(p, \Pi^\mathbb{R}_r(s, t)) \leq \begin{cases} 
\frac{1}{2} & \forall \ p \in \Pi_1^\mathbb{Z}(s, t) \quad \leftarrow \text{naive} \\
\frac{3}{2} & \forall \ p \in \Pi_2^\mathbb{Z}(s, t) \quad \leftarrow \text{standard}
\end{cases}
\]

Discretization Models \textit{(plane & sphere)}

\[ p = (i, j, k) \in \mathbb{Z}^3, \]
\[ X = \{|i|, |j|, |k|\}, \]
\[ h = |i| + |j| + |k|, \]
\[ s = i^2 + j^2 + k^2. \]

\textbf{Theorem (Naive & Standard Spheres)}

\[ S_1 = \left\{ p \in \mathbb{Z}^3 : (r^2 - \max(X) \leq s < r^2 + \max(X)) \right\}. \]
\[ S_2 = \left\{ p \in \mathbb{Z}^3 : r^2 - h \leq s < r^2 + h \right\}. \]
Theorem (Point-to-Sphere Distance)

\[ d_{\perp}(p, S) \leq \begin{cases} \frac{1}{2} & \forall p \in S_1 \quad \leftarrow \text{naive} \\ 2 & \forall p \in S_2 \quad \leftarrow \text{standard} \end{cases} \]
**Symmetry** *(quadraginta octants)*

$#q\text{-octants} = #\text{permutations of } (\pm x, \pm y, \pm z) = 3! \times 2^3 = 48.$
Symmetry (*quadraginta octants*)

\[ r = 10 \]

\[ \#q\text{-octants} = \#\text{permutations of } (\pm x, \pm y, \pm z) = 3! \times 2^3 = 48. \]
Symmetry (quadraginta octants)

#q-octants = #permutations of \((\pm x, \pm y, \pm z)\) = \(3! \times 2^3 = 48\).
Symmetry \((\textit{quadraginta octants})\)

\[
#q\text{-octants} = \#\text{permutations of } (\pm x, \pm y, \pm z) = 3! \times 2^3 = 48.
\]
Here $p = (i, j, k) \in \mathbb{Z}^3$, $s = i^2 + j^2 + k^2$, $X = \{|i|, |j|, |k|\}$.

**Lemma ($S_1$-to-$S$ Distance)**

\[ p \in S_1 \iff d_\perp (p, S) = \left| k - \sqrt{r^2 - (i^2 + j^2)} \right|. \]
Distance bound

Here \( p = (i, j, k) \in \mathbb{Z}^3, s = i^2 + j^2 + k^2, X = \{|i|, |j|, |k|\}. \)

Lemma (Supremum Distance)

\[ p \in S_1 \implies d_{\perp}(p, S) \leq \frac{1}{2}. \]
Here \( p = (i, j, k) \in \mathbb{Z}^3 \), \( s = i^2 + j^2 + k^2 \), \( X = \{|i|, |j|, |k|\} \).

**Lemma (Square Sum Interval)**

\[
p \in S_1 \implies s \in [r^2 - k, r^2 + k - 1].
\]
Here \( p = (i, j, k) \in \mathbb{Z}^3, s = i^2 + j^2 + k^2, X = \{|i|, |j|, |k|\}. \)

**Theorem (Simple Voxel)**

\[
p : d_\perp(p, S) < \frac{1}{2} \text{ is simple/redundant} \iff (s = r^2 + \max(X) - 1) \land (\text{mid}(X) = \max(X)),
\]

where mid denotes the median.
Here \( p = (i, j, k) \in \mathbb{Z}^3 \), \( s = i^2 + j^2 + k^2 \), \( X = \{|i|, |j|, |k|\} \).

**Theorem (Lattice Sphere)**

\[
S_1 = \left\{ p \in \mathbb{Z}^3 : (r^2 - \max(X) \leq s < r^2 + \max(X)) \land ((s \neq r^2 + \max(X) - 1) \lor (\text{mid}(X) \neq \max(X))) \right\}.
\]
Number-theoretic Properties

Lemma (Interval)

The interval $I_n = [(2n - 1)r - n(n - 1), (2n + 1)r - n(n + 1) - 1]$ contains the sum of the squares of $x$- and $y$-coordinates of the voxels of $S_1$ whose $z$-coordinates are $r - n$, for $n \geq 1$.

Lemma (Interval Length)

The lengths of the intervals $I_n$, starting from $I_1$, decrease constantly by 2.
Theorem (Interval Recurrence)

The sum of squares of x- and y-coordinates of voxels lying on $S_1$ and having z-coordinate $r - n$, lies in $I_n := [u_n, v_n := u_n + l_n - 1]$, where

\[ u_n = \begin{cases} 
0 & \text{if } n = 0 \\
 u_{n-1} + l_{n-1} & \text{otherwise}; 
\end{cases} \quad (1) \]

\[ l_n = \begin{cases} 
 r & \text{if } n = 0 \\
 2r - 2 & \text{if } n = 1 \\
  l_{n-1} - 2 & \text{otherwise}. 
\end{cases} \quad (2) \]
First q-octant of the naive sphere of \( r = 23 \).
Theorem (Next Voxel)

If \((i,j,k)\) is the current voxel of \(A_i\), then the next voxel of \(A_i\) is \((i,j+1,k-d)\), where \(d \in \{\emptyset, 0, 1\}\) is given as follows.

<table>
<thead>
<tr>
<th>Interval</th>
<th>(j &lt; k - 1)</th>
<th>(j = k - 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K_0)</td>
<td>0</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>(K_{-1})</td>
<td>1</td>
<td>(\emptyset)</td>
</tr>
</tbody>
</table>

Here, \(d = \emptyset\) implies that there does not exist an appropriate value of \(d\).
Number-theoretic Properties

Theorem (Next Arc)

If \((i, j = i, k)\) is the first voxel of \(A_i\), then the first voxel of \(A_{i+1}\) is \((i + 1, i + 1, k - d)\), where \(d \in \{\emptyset, 0, 1, 2\}\) is given as follows.

<table>
<thead>
<tr>
<th>Interval</th>
<th>(i &lt; k - 2)</th>
<th>(i = k - 2)</th>
<th>(i = k - 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K_0)</td>
<td>0</td>
<td>0</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>(K_{-1})</td>
<td>1</td>
<td>(\emptyset, 1)</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>(K_{-2})</td>
<td>2</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
</tr>
</tbody>
</table>
Algorithm 1: LS3 \((r)\)

1. \(\text{int } i \leftarrow j \leftarrow 0, k \leftarrow k_0 \leftarrow r, s \leftarrow s_0 \leftarrow 0, v \leftarrow v_0 \leftarrow r - 1, l \leftarrow l_0 \leftarrow 2v_0\)
2. \(\text{voxel set } S \leftarrow \{\}\)
3. \(\text{while } i \leq k \text{ do } \triangleright \text{ arc generator}\)
   
   4. \(\quad \text{while } j \leq k \text{ do } \triangleright \text{ voxel generator}\)
   
   5. \(\quad \quad \text{if } s > v \text{ then } \triangleright d = 1 \text{ (Theorem 23)}\)
   
   6. \(\quad \quad \quad k \leftarrow k - 1, v \leftarrow v + l, l \leftarrow l - 2 \triangleright \text{ Theorem 22}\)
   
   7. \(\quad \quad \quad \text{if } ((j \leq k) \land ((s \neq v) \lor (j \neq k)) \text{ then } \triangleright \text{ Lattice Sphere Thm}\)
   
   8. \(\quad \quad \quad S \leftarrow S \cup \{(i', j', k'): \{|i'|\} \cup \{|j'|\} \cup \{|k'|\} = \{i, j, k\}\}\)
   
   9. \(\quad \quad \quad s \leftarrow s + 2j + 1, j \leftarrow j + 1\)
   
10. \(\quad s_0 \leftarrow s_0 + 4i + 2, i \leftarrow i + 1\)
11. \(\quad \text{while } (s_0 > v_0) \land (i \leq k_0) \text{ do } \triangleright \text{ next arc init (Theorem 24)}\)
12. \(\quad \quad k_0 \leftarrow k_0 - 1, v_0 \leftarrow v_0 + l_0, l_0 \leftarrow l_0 - 2 \triangleright \text{ Theorem 22}\)
13. \(\quad j \leftarrow i, k \leftarrow k_0, v \leftarrow v_0, l \leftarrow l_0, s \leftarrow s_0\)
14. \(\text{return } S\)
And so the lattice spheres are produced...
## Techniques

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Principle</th>
<th>PR</th>
<th>PL</th>
<th>IntOp</th>
</tr>
</thead>
<tbody>
<tr>
<td>Montani-Scopigno, 1990 [7]</td>
<td>Incremental</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Andres, 1994 [1]</td>
<td>Incremental</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Andres-Jacob, 1997 [2]</td>
<td>Incremental</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Roget-Sitaraman, 2013 [8]</td>
<td>Incremental</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Toutant <em>et al.</em>, 2013 [9]</td>
<td>Morphology</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Biswas-Bhowmick, 2015 [5]</td>
<td><em>N.T.</em>(^a)</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Biswas-Bhowmick, 2015 [3]</td>
<td><em>N.T.</em>(^b)</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

**PR** = print by run; **PL** = print by layer; **IntOp** = based on integer operations; **N.T.** = based on elementary number-theoretic properties.

\(^a\)From Prima Quadraginta Octant to Lattice Sphere through Primitive Integer Operations, *Theoretical Computer Science* (in press), 2015 (doi: [http://dx.doi.org/10.1016/j.tcs.2015.11.018](http://dx.doi.org/10.1016/j.tcs.2015.11.018))

For a spherical shell $W$, the voxel set is

$$
S = \{ p : 0 \leq d_\perp (p, W) < \frac{1}{2} \}
$$
Demo: \texttt{SPHEREBY48SYM}(r_1 = 7, r_2 = 10)

\[ i = 0, \quad j = 0 \]
Demo: $\text{SPHEREBY48SYM}(r_1 = 7, r_2 = 10)$

$i = 0, j = 1$
Demo: \( \text{SPHEREBY48SYM}(r_1 = 7, r_2 = 10) \)

\[ i = 0, j = 2 \]
Demo: $\text{SPHEREBY48SYM}(r_1 = 7, r_2 = 10)$

$i = 0, j = 3$
Demo: \textbf{SPHEREBY48SYM}(r_1 = 7, r_2 = 10)

\[ i = 0, \quad j = 4 \]
Demo: \textbf{SPHEREBY48SYM}(r_1 = 7, r_2 = 10)

\[ i = 0, \quad j = 7 \]
Demo: $\text{SPHEREBY48SYM}(r_1 = 7, r_2 = 10)$

\[ i = 1, \ j = 4 \]
**Demo:** $\text{SPHEREBY48SYM}(r_1 = 7, r_2 = 10)$

$i = 1, j = 7$
Demo: $\text{SPHEREBY48SYM}(r_1 = 7, r_2 = 10)$

\[ i = 2, j = 7 \]
Demo: $\text{SPHEREBY48SYM}(r_1 = 7, r_2 = 10)$

$i = 3, j = 6$
Demo: $\text{SPHEREBY48SYM}(r_1 = 7, r_2 = 10)$

$i = 4, j = 6$
D emo: \texttt{SPHEREBY48SYM}(r_1 = 7, r_2 = 10)

End
Demo: \textsc{LayerTheSphere}(r_1 = 7, r_2 = 10)

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**Demo:** \textsc{LayerTheSphere}(r_1 = 7, r_2 = 10)

\[ k = -9 \]
Demo: \textsc{LayerTheSphere}(r_1 = 7, r_2 = 10)

\[ k = -8 \]
Demo: \textsc{LayerTheSphere}(r_1 = 7, r_2 = 10)

\[ k = -7 \]
**Demo:** LayerTheSphere\((r_1 = 7, r_2 = 10)\)

\[ k = -6 \]
Demo: LayerTheSphere\((r_1 = 7, r_2 = 10)\)

\[ k = -5 \]
Demo: \textsc{LayerTheSphere} \((r_1 = 7, r_2 = 10)\)

\[
k = -4
\]
**Demo:** \textsc{LayerTheSphere}(r_1 = 7, r_2 = 10)

\[ k = -3 \]
Demo: $\text{LAYER\ THE\ SPHERE}(r_1 = 7, r_2 = 10)$

$$k = -2$$
Demo: \textsc{LayerTheSphere}(r_1 = 7, r_2 = 10)

$k = 10$ (end)
Spherical Geodesics
Spherical Geodesics [4]

\[ r = 12, \ s = (10, -2, 6) \in Q_{15}, \ t = (-3, 10, 6) \in Q_{12}. \]
Naive sphere, Standard plane \( \implies \) Class NS \((l = 1)^a\)

\(^a\) R. Biswas and P. Bhowmick, On different topological classes of spherical geodesic paths and circles in \(\mathbb{Z}^3\), *Theoretical Computer Science* 605:146–163, 2015.
DSGP Topological Classes

Naive model

Standard model

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<table>
<thead>
<tr>
<th>Class</th>
<th>NN (1,1)</th>
<th>NS (1,2)</th>
<th>SN (2,1)</th>
<th>SS (2,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example</td>
<td>$r = 17$, $s = (-6, -1, 16)$, $t = (2, 14, 10)$</td>
<td>Naive-naive ($m = 1$, $n = 1$)</td>
<td>Naive-standard ($m = 1$, $n = 2$)</td>
<td>Standard-naive ($m = 2$, $n = 1$)</td>
</tr>
</tbody>
</table>

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From circle to sphere and to related problems in the digital space
DSGP Topological Classes

From circle to sphere and to related problems in the digital space

- Classes
- Geodesics
- Conclusion
- Ref
DSGP Topological Classes

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DSGP Topological Classes

**Theorem (Class Bounds)**

The respective upper bounds of the isothetic distance of the DSGP \( \pi_{m,n}^{(l)}(s,t) \) from the real sphere \( S \) and of that from the real plane \( \Pi_r^{\mathbb{R}}(s,t) \) for classes NS, SN, and SS are as follows.

\[
\begin{align*}
\max_{p \in \pi_{m,n}^{(l)}(s,t)} \{ d_{\perp}(p, S) \} & \begin{cases} < \frac{1}{2} & \text{if } l \in \{0, 1\}, m = 1, n = 2 \\ \leq 2 & \text{if } l \in \{0, 1\}, m = 2, n = 1 \\ \leq 2 & \text{if } l \in \{0, 1, 2\}, m = 2, n = 2 \end{cases} \\
\max_{p \in \pi_{m,n}^{(l)}(s,t)} \{ d_{\perp}(p, \Pi_r^{\mathbb{R}}(s,t)) \} & \begin{cases} \leq \frac{3}{2} & \text{if } l \in \{0, 1\}, m = 1, n = 2 \\ \leq \frac{1}{2} & \text{if } l \in \{0, 1\}, m = 2, n = 1 \\ \leq \frac{3}{2} & \text{if } l \in \{0, 1, 2\}, m = 2, n = 2 \end{cases}
\end{align*}
\]
DSGP Topological Classes

\[ NS \ (l = 0) : 16 \]
DSGP Topological Classes

NS \((l = 1) : 19\)
DSGP Topological Classes

Different classes of DSGP (red voxels)

$$(r = 17, s = (-6, -1, 16), t = (2, 14, 10))$$
From circle to sphere and to related problems in the digital space

\[ \text{NS}(l = 1), r = 30 \]
DSGP Topological Classes

\[ NS(l = 0), r = 30 \]
Open Problems

- **Discrete 3D circles**—maximum symmetry + minimum length-and-deviation.

NS \((0, 1, 2)\) : 90 vox

NS \((1, 1, 2)\) : 102 vox
Open Problems

- **Voxel strengthening**—for improved 3D printing, by reshaping voxel as truncated tetrahedron, octahedron, sphere, or even Great Invention Kit (GIKs) [6].
A microscopic view of rounded crystals produced by the scientists for 3d-printing
Open Problems

- *iso-contours, geodesic distance query*—as in 3D real space [10, 11].
- *Rational specification*—characterization and algorithm.
Further Reading

Further Reading

- R. Biswas and P. Bhowmick.  
  From prima quadraginta octant to lattice sphere through primitive integer operations.  

- J. Hiller and H. Lipson.  
  Design and analysis of digital materials for physical 3D voxel printing.  

- C. Montani and R. Scopigno.  
  Graphics gems (Chapter: *Spheres-to-voxels conversion*), A. S. Glassner (Ed.).  

- B. Roget and J. Sitaraman.  
  Wall distance search algorithm using voxelized marching spheres.  
Further Reading

Digital circles, spheres and hyperspheres: From morphological models to analytical characterizations and topological properties.

S.-Q. Xin, X. Ying, and Y. He.
Constant-time all-pairs geodesic distance query on triangle meshes.

X. Ying, X. Wang, and Y. He.
Saddle Vertex Graph (SVG): A novel solution to the discrete geodesic problem.
Thank You