Closure Properties of CFL

25 Mar 2019

Intruction : Write the answers to the problems neatly in loose sheets with your name and roll number. Submit to the TA in the subsequent class.

1. Prove that, CFL's are not closed under Complementation. **(In class)**

   Solution: We know the CFL's are closed under union. If they were closed under complementation, they would, by DeMorgan's law, $L \cap R = \overline{\overline{L} \cup \overline{R}}$ be closed under intersection. But we know that CFLs are not closed under intersection.

2. Design a suitable unambiguous grammar for arithmetic expressions having operators $^*, /, +, -$ and positive integers as operands with usual precedence relation. Extend this grammar with operators $=, =, \leq, \geq$ which are all having same level of precedence. Also the precedence of these relational operators is below that of $+, -$. This implies, $1 + 2 \leq 3$ is equivalent to $(1 + 2) \leq 3$. Also note that the relational operators are non-associative, i.e. $1 \leq 2 \leq 3 + 4$ is not a legal expression. **(In class)**

3. Let L be a CFL and $LEAST(L) = \{x \in L | \texttt{no proper prefix of } x \in L\}$. Show that CFLs are not closed under $LEAST$ operation. **(In class)** (Hint: We have to find out a counterexample. Start with $L = \{0^i 1^j 2^k | i \leq k \; or \; j \leq k\}$. Show that $LEAST(L)$ is not CFL.)

   Solution: Let L be the CFL $\{0^i 1^j 2^k | i \leq k \; or \; j \leq k\}$. L is generated by the CFG,

   $S \rightarrow AB|C, \;\; A \rightarrow 0A|\epsilon, \;\; C \rightarrow 1B2|B2|\epsilon, \;\; C \rightarrow 0C2|C2|D, \;\; D \rightarrow 1D|\epsilon$

   $LEAST(L) = \{0^i 1^j 2^k | k = min(i, j)\}$. We claim $LEAST(L)$ is not a CFL. Suppose it were, and let n be the pumming lemma constant. Consider $z = 0^n 1^n 2^n = uvwxy$. If vx contains no $2's$, then uwy is not in $LEAST(L)$. If vx has a 2, it cannot have a 0, since $|vwx| \leq n$. Thus uv^2wx^2y has at least $n + 1$ 2's, at least n 1's and exactly n 0's; it is thus not in $LEAST(L)$.

4. A linear language is a CFL with all productions having atmost one non-terminal in the R.H.S. Prove the following pumping lemma for linear

languages - “If L is a linear language then there is a constant n such that if $z \in L$ and $|z| \geq n$ then $\exists u, v, w, x, y$ such that $z = uvwxy$, $|uvxy| \leq n$, $|vx| \geq 1$ and $\forall i \geq 0$, $uv^iwx^iy \in L$.”

Solution: Since the language is linear there exists a linear grammar G for it. For the argument it is convenient to assume that G has no unit-productions and no λ-productions. Consider now the derivation of a string $w \in L(G)$

$$S \stackrel{*}{\Longrightarrow} uAz \stackrel{*}{\Longrightarrow} uvAyz \stackrel{*}{\Longrightarrow} uvxyz \stackrel{*}{\Longrightarrow} w.$$

Assume, for the moment, that for every $w \in L(G)$, there is a variable A, such that

(a) in the partial derivation $S \stackrel{*}{\Longrightarrow} uAz$ no variable is repeated

(b) in the partial derivation $S \stackrel{*}{\Longrightarrow} uAz \stackrel{*}{\Longrightarrow} uvAyz$ no variable except A is repeated

(c) the repetition of A must occur in the first m steps, where m can depend on the grammar, but not on w.

If this is true, then the lengths of u, v, y, z must be bounded independent of w. This in turn implies that the conditions of the lemma must hold. To complete the argument, we must still demonstrate that the above conditions hold for every linear grammar. This is not hard to see if we look at sequences in which the variables can occur.

5. Show that $\{a^ib^ic^jd^j \mid i \geq 1, j \geq 1\}$ is not a linear language.

6. Show that if L is a CFL over a one-symbol alphabet, then L is regular. **(Home)**
Solution: If the language L is finite, there is nothing to prove.
Consider an infinite language L and let the pumping constant be k. So each w of length $\geq k$ in the language can be written as $w = uvxyz$ so that $|vxy| \leq k, |vy| > 0$ and for all $i \geq 0, uv^ixy^iz \in L$. As there is only one alphabet, we write the last clause as $uxz(vy)^* \subseteq L.|vy| = p$, so $uxz(a^p)^i \in L$ for all $i \geq 0$. Let $\alpha = k!$, we claim that $w(a^\alpha)^m \in L$ for all $m \geq 0$, as $\alpha \times m = p \times \frac{m \times \alpha}{p}$. Note that α does not depend on w. So for each word $w \in L$ and $|w| \geq k, w(a^\alpha)^m \in L$, for all $m \geq 0$. We observe that each $w \in L$ and $|w| \geq k$ is an element of $a^{k+i}(a^\alpha)^*$ where $0 \leq i < \alpha$. Consider the least element w_i of $L \cap a^{k+i}(a^\alpha)^*$. So the language $L = L_1 \cup L_2$, where L_1 is the finite collection of strings of length $< k$ and $L_2 = \bigcup_{0 \leq i < \alpha} w_i(a^\alpha)^*$. So L is regular.