CS21004 - Tutorial 10

Solution Sketch

Instructions: For the problems with 'to submit', please write the answers neatly in loose sheets and submit to the TA before the end of the tutorial.

1. Let $max(L) = \{w | w \in L$ but for no string $wx (x \neq \epsilon)$ is in $L\}$. Are the CFL's closed under the max operation?

   Solution: No.

   Take $L = \{a^i b^j c^k | i \geq k$ or $j \geq k, i, j, k > 0\}$. L is a CFL.

   $Max(L) = \{a^i b^j c^k | k = max(i, j), i, j, k > 0\}$. We can prove that this is not a CFL using Pumping Lemma.

2. Prove or disprove. Let C be a context-free language and R be a regular language. Then $C - R$ is neccessarily context-free, and so is $R - C$. (To submit)

   Solution: $C - R = C \cap \bar{R}$. Since regular languages are closed under intersection, and context free languages are closed under intersection with regular languages $C - R$ is necessarily context-free.

   However, $R - C$ need not be context-free. For a counter example, take $R = \Sigma^*$, we know that context free languages are not closed under complement. E.g., take C to be complement of ww.

3. Let $half(L) = \{w |$ for some x such that $|x| = |w|, wx \in L\}$. Notice that odd-length words in L do not contribute to $half(L)$. Are the CFLs closed under *half* operation? (To submit)

   Solution: No. Take $L = \{a^n b^n c^m d^{3m} | m, n > 0\}$. Clearly, this is CFL. Now, take $half(L) \cup a^* b^* c^+$, this will give only the cases where $3m < 2n + m$, i.e., $m < n$.

   Now, take $L' = \{a^n b^n c^j | j < n\}$. Clearly, this is not CFL (can prove using pumping lemma).

4. A *shuffle* of two strings α and β is a string γ of length $|\alpha| + |\beta|$, in which α and β are non-overlapping subsequences (not neccessarily substrings). For example, all shuffles of *ab* and *cd* are *abcd*, *cabd*, *cdab*, *acbd*, *acdb* and *cadb*. For two languages A and B, we define $shuffle(A, B)$ as the language consisting of all shuffles of $\alpha \in A$ and all $\beta \in B$. Prove or disprove the following statements.

(a) If L is a CFL and R is a regular language then $shuffle(L, R)$ is a CFL.

Solution: Yes, we can construct a PDA for $shuffle(L, R)$. On each read of the input tape, the machine guesses whether the input came from L or R, and transitions accordingly.

$Q = Q_L \times Q_R$ ($\delta([p\ q], a, \beta)$ contains $\{([r\ q], \gamma), ([p\ t], \beta)\}$ if $\delta_L(p, a, \beta)$ contains (r, γ) and $\delta_R(q, a) = t$. We accept if one of the branches ends in a final state for L and a final state for R.

(b) If L_1 and L_2 are CFLs then $shuffle(L_1, L_2)$ is a CFL.

Solution: No.

Take $L_1 = \{a^n b^n | n > 0\}$ and $L_2 = \{c^m d^m | m > 0\}$. Now, $shuffle(L_1, L_2) \cap a^* c^* b^* d^*$ will give us all the strings of the form $\{a^n c^m b^n d^m | m, n > 0\}$, which is clearly not a CFL.

(c) Consider $L = \{a^n b^n c^n | n \geq 0\}$. Is $\overline{L}$ a CFL? (Home)

Solution: Yes. $\overline{L} = \overline{a^* b^* c^*} \cup \{a^i b^j c^k | i \neq j \text{ or } j \neq k \text{ or } i \neq k\}$

The first is a regular language, and second is a union of three context free languages, hence context free. The intersection of context free language with regular language is closed.