Induction on Strings

7 Jan 2019

Instruction : Write the answers to the problems neatly in loose sheets with your name and roll number. Submit to the TA at the end of the class.

1. A palindrome can be defined as a string that reads the same forward and backward, or by the following definition.

   (a) ϵ is a palindrome.

   (b) If a is any symbol, then the string a is a palindrome.

   (c) If a is any symbol and x is a palindrome, then axa is a palindrome.

   (d) Nothing is a palindrome unless it follows from (a) through (c).

   Prove by induction that the two definitions are equivalent.

   Solution : Let the definition provided here be called $def1$ while the usual definition of palindromes be termed as $def2$. Usual definition means that the string reads the same both forward and backward. Both definitions are equivalent implies, both definitions capture exactly the same subset of Σ^*. We prove this by induction on string length, the base case being the empty string.

   For ϵ, it is part of $def1$ (clause 1) while it trivially satisfies $def2$. Similar argument holds for strings of unit length (clause 2 in $def1$). For length 2 palindromes, they satisfy $def1$ being of the type aa with $x = \epsilon$ (clause 3). Strings of type aa also satisfy $def2$ being the same symbol repeated twice. Now let us assume both definitions to be equivalent upto strings of length $n > 2$ in Σ^*.

   Consider a string σ with $|\sigma| = n + 1$ which is palindrome as per $def2$. That implies $\sigma = \sigma^R$ (applying $def2$). Hence it must be the case that σ starts and ends with same symbol. Hence $\exists \sigma' \in \Sigma^*$, $a \in \Sigma$ such that $\sigma = a\sigma'a$. Also, $\sigma = \sigma^R \Rightarrow a\sigma'a = (a\sigma'a)^R \Rightarrow a\sigma'a = a{\sigma'}^R a \Rightarrow \sigma' = {\sigma'}^R$. Thus σ' is palindrome as per $def2$. Since $|\sigma'| = n - 1$ and $def1$, $def2$ are equivalent for string length upto n, we have σ' as palindrome also for $def1$. Now, applying clause 3 of $def1$, we have $\sigma = a\sigma'a$ as palindrome (as per $def1$).

   Consider a string σ with $|\sigma| = n + 1$ which is palindrome as per $def1$. Since $n > 2$, we must have a palindrome x such that $|x| = n - 1$ and

$axa = \sigma$ for some symbol a. x should satisfy $def2$ and hence $x = x^R$. So, $\sigma^R = (axa)^R = ax^Ra = axa = \sigma$. This σ is also palindrome as per $def2$.

2. The strings of balanced parenthesis can be defined in at least two ways.

   (a) A string w over alphabet $\{(,)\}$ is balanced if and only if:

      i. w has an equal number of ('s as)'s, and

      ii. any prefix of w has at least as many ('s as)'s.

   (b) i. ϵ is balanced.

      ii. If w is a balanced string, then (w) is balanced.

      iii. If w and x are balanced strings, then so is wx.

      iv. Nothing else is a balanced string.

   Prove by induction on the length of a string that definitions (a) and (b) define the same class of strings.

   Solution : Let P_Σ^n be the set of balanced parenthesis upto length $2n$. $P_\Sigma^0 = \{\epsilon\}$ which is trivially satisfying def1 and def2. Let def1 and def2 agree upto P_Σ^n. Now $P_\Sigma^{n+1} = P_\Sigma^n \cup X$ with X being the set of balanced parenthesis of length exactly $2n + 2$.

   Let $x \in X$ be a balanced parenthesis satisfying def1. Consider the prefix of x of length 1. As per condition 2 of def1, it has to be '('. (If the first symbol is ')', condition 2 is not satisfied.) As per condition 1 of def1, x has $n + 1$ '(' and $n + 1$ ')'. We now argue that the last symbol of x has to be ')'. Otherwise, if $x = x_1($, then x_1 is a prefix with $n + 1$ ')' and n '(' (violates condition 2). Hence, as per def1 $x = (w)$ (has to start and end with '(' and ')' respectively). There are two options now.

   (a) w is a balanced string of length $2n$ as per def1. Then w is a balanced string also as per def2 (the definitions agree upto length 2n). Then x satisfies def2 (clause2).

   (b) w is not a balanced string as per def1. This can only happen if clause 2 of def1 is violated by w (clause 1 is satisfied). Let w_1 be the smallest such prefix of w with less '(' than ')'. Note that '$(w_1$' manages to be a prefix with at least as many '(' as ')'. Hence, in w_1 there is exactly one '(' less. Thus '$(w_1$' satisfies both conditions of def1 (condition 2 is guaranteed by w_1 being the smallest possible violator). Thus $(w_1$ is a balanced string as per def1 (and hence def2). With $x = (w) = (w_1w_2)$, what about w_2 ? Since condition 1 and 2 of def1 are satisfied by both $(w_1$ and w, they are also satisfied by $w_2)$. Still let us show that. Surely, $w_2)$ has same number of '(' and ')' since both (w) and $(w_1$ are balanced. Consider any prefix σ of $w_2)$. Note, $(w_1\sigma$ has at least as many '(' as ')'. $(w_1$ has same number of '(' and ')'. Hence σ has at least as many '(' as ')'.

   Both $(w_1$ and $w_2)$ are balanced as per both defs. Hence $x = (w)$ is balanced also as per def2 (clause3).

Assume def2 and prove the reverse now.

3. Prove that any equivalence relation R on a set S partitions S into disjoint equivalence classes.

   Solution :
   Let, $x, y \in S$ and $[[x]]_R \cap [[y]]_R \neq \phi$ and suppose $z \in [[x]]_R$, $z \in [[y]]_R$
   Hence by definition of Equivalence Class, $(x, z) \in R$, $(y, z) \in R$
   Let $c \in [[x]]_R$ i.e. $(x, c) \in R$
   By definition of Equivalence relation, R is symmetric. So, $(z, x) \in R$
   By definition of equivalence relation, R is transitive. So,
   $(z, x) \in R \wedge (x, c) \in R \Rightarrow (z, c) \in R$ and $(y, z) \in R \wedge (z, c) \in R \Rightarrow (y, c) \in R$
   This gives $c \in [[y]]_R$ and hence, $[[x]]_R \subseteq [[y]]_R$

   Considering $c \in [[y]]_R$, it can be proved that $[[y]]_R \subseteq [[x]]_R$ in similar way.
   $[[x]]_R \subseteq [[y]]_R \wedge [[y]]_R \subseteq [[x]]_R \Rightarrow [[x]]_R = [[y]]_R$
   Thus, $[[x]]_R \cap [[y]]_R \neq \phi \Rightarrow [[x]]_R = [[y]]_R$.
   Following the above results, we can say, equivalence classes are either same or disjoint. Hence any equivalence relation R on a set S partitions S into disjoint equivalence classes.

4. Show that the following are equivalence relations and give their equivalence classes.

   (a) R_1 on integers $\rightarrow iR_1j$ iff $i = j$.

   (b) R_2 on people $\rightarrow pR_2q$ iff p and q were born on the same hour of same day of some year.

   (c) In (b) replace "some year" with "same year".

   Solution :

   (a) R_1 is a relation on set of integers $\mathbb{Z}$ such that $iR_1j \implies i = j$

      i. **Proof of Reflexivity**: Let, $a \in \mathbb{Z}$
         As we can say $a = a \implies aR_1a$

      ii. **Proof of Symmetry**: Let, $a, b \in \mathbb{Z}$ such that $a = b \implies aR_1b$
         From $a = b$ we can say $b = a \implies bR_1a$

      iii. **Proof of Transitivity**: Let, $a, b, c \in \mathbb{Z}$ such that $a = b, b = c \implies aR_1b,\ bR_1c$
         As we can say $a = c \implies aR_1c$

      Clearly this R_1 will divide $\mathbb{Z}$ into as many equivalent classes as many integers are present in Integer set.

   (b) R_2 is a relation on set of people $\mathcal{P}$ such that $pR_2q \implies p$ and q were born on the same hour of same day of some year.

      i. **Proof of Reflexivity**: Let, $p \in \mathcal{P}$
         Then we can say p and p were born on the same hour of same day of some year. $\implies pR_1p$

ii. **Proof of Symmetry**: Let, $p, q \in \mathcal{P}$ and pR_2q
Then we can say q and p were born on the same hour of same day of some year. $\implies qR_1p$

iii. **Proof of Transitivity**: Let, $p, q, r \in \mathcal{P}$ and pR_2q, qR_2r
As $pR_2q \implies p$ and q were born on the same hour of same day of some year and $qR_2r \implies q$ and r were born on the same hour of same day of some year.then we can clearly say, p and r were born on the same hour of same day of some year. $\implies pR_1r$

Clearly this R_2 will divide $\mathcal{P}$ into 365*24 (366*24, for leap years) equivalence classes i.e. the number of hours in any year.