CS21201 Discrete Structures
Practice Problems + Tutorial Solutions
Recurrence Relations

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1. How many lines are printed by the call $f(n)$ for an integer $n \geq 0$?

```
void f ( int n )
{
    int m;
    printf("Hi\n");
    m = n - 1;
    while (m >= 0) { f(m); m -= 2; }
}
```

Suppose, L_n be the number of lines printed by the call $f(n)$. So,

$$L_n = \begin{cases} 1 + \sum_{k=0}^{(n-1)/2} L_{2k}, & \text{if } n \text{ is odd} \\ 1 + \sum_{k=0}^{(n-2)/2} L_{2k+1}, & \text{if } n \text{ is even} \end{cases} \quad \text{for } n \geqslant 0$$

Here, for both cases we derive, $L_n - L_{n-2} = L_{n-1}$ with $L_0 = 1,\ L_1 = 2$.

This is the same recurrence as we got in Fibonnaci series computation having $L_n = F_{n+1}$. Therefore,

$$L_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right].$$

2. Solve for the following divide-and-conquer recurrence:

$$T(n) = 2T(n/2) + \frac{n}{\log n} \text{ with } T(1) = 1.$$

Dividing both the sides of the given recurrence by n, we obtain:

$$\frac{T(n)}{n} = \frac{T(n/2)}{n/2} + \frac{1}{\log n}$$

Assuming $n = 2^k$ and $S(k) = \frac{T(n)}{n} = \frac{T(2^k)}{2^k}$, we can rewrite the above as:

$$S(k) = S(k-1) + \frac{1}{k} \quad \text{with } S(0) = 1$$

Now, if we recursively simplify the righthand side of the equation above, we get:

$$\begin{aligned} S(k) &= S(k-1)+\frac{1}{k} \\ &= S(k-2)+\frac{1}{k-1}+\frac{1}{k} \\ &= S(k-3)+\frac{1}{k-2}+\frac{1}{k-1}+\frac{1}{k} \\ &= \cdots = S(0)+\sum_{i=1}^{k}\frac{1}{i} \end{aligned}$$

$$\therefore T(n) = nS(k) = n\left[1+\sum_{i=1}^{\log n}\frac{1}{i}\right].$$

3. Let $a_n = 7a_{n/2} - 6a_{n/4}$, $a_1 = 2$, $a_2 = 7$. Find a_n.

Take $n = 2^m$

$$a_{2^m} = 7a_{2^{m-1}} - 6a_{2^{m-2}}$$

$$a_{2^m} = b_m$$

$$\Rightarrow b_m = 7b_{m-1} - 6b_{m-2},\ b_0 = 2,\ b_1 = 7$$

Characteristic equation is

$$x^2 - 7x + 6 = 0 \Rightarrow x = 6, 1$$

$$\Rightarrow b_m = \lambda 6^m + \mu 1^m$$

For $m = 0,\ 2 = \lambda + \mu$

For $m = 1,\ 7 = 6\lambda + \mu$

$$\Rightarrow \lambda = 1, \mu = 1$$

$$\Rightarrow b_m = 6^m + 1$$

$$\Rightarrow a_{2^m} = 6^m + 1$$

$$\Rightarrow a_n = 6^{\log_2 n} + 1$$

$$\Rightarrow a_n = n^{\log_2 6} + 1$$

4. Let $a_n, n \geq 0$ be the count of strings over $\{0, 1, 2\}$ containing no consecutive 1's and no consecutive 2's. Find a recurrence relation for a_n and solve it.

Let b_n, c_n, d_n, $n \geqslant 1$, denote the counts of the strings of the desired form that start with $0, 1, 2$, respectively. Let us also take $b_0 = 1$, $c_0 = 0$ and $d_0 = 0$. We have the following equations involving these.

$$\begin{aligned}
a_n &= b_n + c_n + d_n \text{ for all } n \geqslant 0, \\
b_n &= a_{n-1} \text{ for all } n \geqslant 1, \\
c_n &= b_{n-1} + d_{n-1} = a_{n-1} - c_{n-1} \text{ for all } n \geqslant 1, \\
d_n &= b_{n-1} + c_{n-1} = a_{n-1} - d_{n-1} \text{ for all } n \geqslant 1.
\end{aligned}$$

Adding the last three equations gives

$$a_n = 3a_{n-1} - (c_{n-1} + d_{n-1}) = 3a_{n-1} - (a_{n-1} - b_{n-1}) = 2a_{n-1} + b_{n-1} = 2a_{n-1} + a_{n-2}$$

for all $n \geqslant 2$. The initial conditions are $a_0 = 1$, $a_1 = 3$. The characteristic equation of the sequence is $r^2 - 2r - 1 = 0$. The roots of the characteristic equation are $1 + \sqrt{2}, 1 - \sqrt{2}$.

So the general solution of this recurrence is of the form

$$a_n = A(1 + \sqrt{2})^n + B(1 - \sqrt{2})^n.$$

The initial conditions give

$$\begin{aligned}
a_0 &= 1 = A + B, \\
a_1 &= 3 = A(1 + \sqrt{2}) + B(1 - \sqrt{2}).
\end{aligned}$$

Solving gives $A = \frac{1+\sqrt{2}}{2}$ and $B = \frac{1-\sqrt{2}}{2}$.

Therefore, the final solution is:

$$a_n = \frac{1}{2}\left[(1 + \sqrt{2})^{n+1} + (1 - \sqrt{2})^{n+1}\right] \text{ for all } n \geqslant 0.$$

5. Let $a_n, n \geq 1$ satisfy $a_1 = 1$ and

$$a_n = \begin{cases} 2a_{n-1} & \text{if } n \text{ is odd} \\ 2a_{n-1} + 1 & \text{if } n \text{ is even} \end{cases} \text{ for } n \geqslant 2$$

Develop a recurrence relation for a_n that holds for both odd and even n, and solve it.

For both cases, $a_n - a_{n-2} = 2(a_{n-1} - a_{n-3}) \Rightarrow a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$. Moreover, $a_2 = 3$ and $a_3 = 6$.

The characteristics equation is, $r^3 - 2r^2 - r + 2 = 0$. Solving this, we get the three roots as, $-1, 1, 2$.

So, the general recurrence form is, $a_n = \alpha.(-1)^n + \beta.1^n + \gamma.2^n$.

Now, $a_1 = -\alpha + \beta + 2\gamma = 1$, $a_2 = \alpha + \beta + 4\gamma = 3$ and $a_3 = -\alpha + \beta + 8\gamma = 6$.

Solving these, we get, $\alpha = \frac{1}{6}$, $\beta = -\frac{1}{2}$, and $\gamma = \frac{5}{6}$.

Therefore, $a_n = \frac{1}{6}.(-1)^n - \frac{1}{2}.1^n + \frac{5}{6}.2^n = \frac{1}{6}.\left[5.2^n + (-1)^n - 3\right]$.

6. Solve the following recurrence relation and deduce the closed-form expression for $T(n)$.

$$T(n) = \begin{cases} \sqrt{n}T(\sqrt{n}) + n(\log_2 n)^d, & \text{if } n > 2 \\ 2 & \text{if } n = 2 \end{cases} \quad (d \geqslant 0)$$

Given that $T(n) = \sqrt{n}T(\sqrt{n}) + n\log_2^d n$ (where $d \geqslant 0$) and $T(2) = 2$, we have:

$$\begin{aligned}
\frac{T(n)}{n} &= \frac{T(\sqrt{n})}{\sqrt{n}} + \log_2^d n && \ldots\left[\textit{dividing both sides by } n\right] \\
\Rightarrow\ S(n) &= S(\sqrt{n}) + \log_2^d n && \ldots\left[\textit{assuming } S(n) = \frac{T(n)}{n}\right] \\
\Rightarrow\ S(2^{2^k}) &= S(2^{2^{(k-1)}}) + (2^k)^d && \ldots\left[\textit{substituting } n = 2^{2^k}\right] \\
\Rightarrow\ R(k) &= R(k-1) + (2^k)^d && \ldots\left[\textit{let } R(k) = S(2^{2^k})\right] \\
\Rightarrow\ R(k) &= R(0) + (2^d)^1 + (2^d)^2 + \cdots + (2^d)^{k-1} + (2^d)^k && \ldots\left[\textit{because } (2^k)^d = 2^{kd} = (2^d)^k\right] \\
\Rightarrow\ R(k) &= 1 + \sum_{i=1}^{k} (2^d)^i && \ldots\left[S(2) = \frac{T(2)}{2} = 1,\ \textit{implying } R(0) = S(2^{2^0}) = 1\right] \\
\Rightarrow\ R(k) &= \begin{cases} \frac{(2^d)^{(k+1)}-1}{2^d-1}, & \text{if } d > 0 \\ 1+k, & \text{if } d = 0 \end{cases}
\end{aligned}$$

$$\therefore\ R(k) = S(2^{2^k}) = \begin{cases} \frac{(2^d)^{(k+1)}-1}{2^d-1}, & \text{if } d > 0 \\ 1+k, & \text{if } d = 0 \end{cases}, \quad \text{where } n = 2^{2^k} \text{ and } S(n) = \frac{T(n)}{n}.$$

Finally,

$$S(n) = \begin{cases} \frac{2^d \log_2^d n - 1}{2^d-1}, & \text{if } d > 0 \\ 1 + \log_2\log_2 n, & \text{if } d = 0 \end{cases} \quad \Rightarrow \quad T(n) = \begin{cases} \frac{2^d n\log_2^d n - n}{2^d-1}, & \text{if } d > 0 \\ n + n\log_2\log_2 n, & \text{if } d = 0 \end{cases}$$

7. Deduce the running times of divide-and-conquer algorithms in the big-Θ notation if their running times satisfy the following relations:
   a. $T(n) = T(2n/3) + T(n/3) + n\,log\,n$
   b. $T(n) = T(n/5) + T(7n/10) + n$

(a) This recurrence is not in the standard form described earlier, but we can still solve it using recursion trees. Now nodes in the same level of the recursion tree have different values, and different leaves are at different levels. However, the nodes in any complete level (that is, above any of the leaves) sum to $\approx n\log n$. Moreover, every leaf in the recursion tree has depth between $\log_3 n$ and $\log_{3/2} n$. To derive an upper bound, we overestimate $T(n)$ by ignoring the base cases and extending the tree downward to the level of the deepest leaf. Similarly, to derive a lower bound, we overestimate $T(n)$ by counting only nodes in the tree up to the level of the shallowest leaf. These observations give us the upper and lower bounds.

$$\begin{aligned}
(c_1.n\log n)(\log_3 n) \ &\leqslant\ T(n)\ \leqslant\ (c_2.n\log n)(\log_{3/2} n) \\
(c_1.n\log n)\left(\frac{\log n}{\log 3}\right) \ &\leqslant\ T(n)\ \leqslant\ (c_2.n\log n)\left(\frac{\log n}{\log 3 - 1}\right)
\end{aligned}$$

Since these bounds differ by only a constant factor, we have $T(n) = \Theta(n\log^2 n)$.

(b) Again, we have a lopsided recursion tree. If we look only at complete levels of the tree, we find that the level sums form a descending geometric series,

$$T(n) = n + \frac{9n}{10} + \frac{81n}{100} + \cdots \quad \Rightarrow \quad n \leqslant T(n) \leqslant 10n$$

We can get an upper bound by ignoring the base cases entirely and growing the tree out to infinity, and we can get a lower bound by only counting nodes in complete levels. Either way, the geometric series is dominated by its largest term, so $T(n) = \Theta(n)$.

8. Pell numbers are defined as $P_0 = 0$, $P_1 = 2$, $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$.
   a. Deduce a closed-form formula for P_n.

b. Prove that $\begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n$ for all $n \geq 1$.

c. Prove that $\lim_{n\to\infty} \frac{P_{n-1}+P_n}{P_n} = \sqrt{2}$.

d. Prove that if P_n is prime, then n is also prime.

Solution:

a.) The recurrence relation is $P_n = 2P_{n-1} + P_{n-2}$ with $P_0 = 0,\ P_1 = 2$.

The characteristic equation is thus $r^2 - 2r - 1 = 0$ with roots $r = 1 + \sqrt{2},\ r = 1 - \sqrt{2}$. General solution is of the form $P_n = A(1 + \sqrt{2})^n + B(1 - \sqrt{2})^n$.

Using $P_0 = 0$ and $P_1 = 2$, we have:

$A = \frac{1}{\sqrt{2}}$ and $B = \frac{-1}{\sqrt{2}}$

Thus, $P_n = \frac{1}{\sqrt{2}}((1 + \sqrt{2})^n - (1 - \sqrt{2})^n)$.

b.) Use induction to prove. Base case (n = 1) is trivial from the given conditions. Assume the induction holds for some $k,\ k \geq 1$. Use $P_{k+2} = 2P_{k+1} + P_k$ and $P_{k+1} = 2P_k + P_{k-1}$ to prove for $k + 1$.

c.) The closed-form formula can be used here. As n grows, the $(1 - \sqrt{2})^n$ term becomes negligible in both the numerator and the denominator (will grow asymptotically like the dominant root $(1 + \sqrt{2})^n$). Thus, we can write:

$$P_n \approx A(1 + \sqrt{2})^n \text{ for some } A \text{ depending on the initial condition}$$

Or, $$\frac{P_{n-1}+P_n}{P_n} \approx \frac{A(1+\sqrt{2})^{n-1}+A(1+\sqrt{2})^n}{A(1+\sqrt{2})^n}$$

$$\lim_{n\to\infty} \frac{A(1+\sqrt{2})^{n-1}+A(1+\sqrt{2})^n}{A(1+\sqrt{2})^n} = \frac{1}{1+\sqrt{2}} + 1 = \sqrt{2}$$

d.) You can prove this by showing that if d is a divisor of n then P_d is a divisor of P_n. Consider $\alpha = (1 + \sqrt{2})$ and $\beta = (1 - \sqrt{2})$. Let $n/d = k$. Then, P_n can be represented as follows (based on the closed-form formula):

$$P_n = c(\alpha^n - \beta^n), \text{ where } c = \frac{1}{\sqrt{2}} \text{ as derived before}$$

Similarly, $P_d = c(\alpha^d - \beta^d)$

Let $\alpha^d = x$ and $\beta^d = y$,

Now, $\frac{P_n}{P_d} = \frac{x^k - y^k}{x-y}$

It can be shown that $(x - y)$ divides $(x^k - y^k)$, thus proving the claim given.

9. Consider a non-homogeneous recurrence of the form:

$$a_n = c_{k-1}a_{n-1} + c_{k-2}a_{n-2} + \ldots + c_0a_{n-k} + p_1(n)s_1^n + p_2(n)s_2^n.$$

Here, $c_{k-1}, c_{k-2},..., c_0$ are constants (with $c_0 \neq 0$), $p_1(n)$ and $p_2(n)$ are non-zero polynomials in n, and s_1, s_2 are distinct non-zero constants. Propose a method to solve this recurrence.

Consider two new sequences u_n and v_n satisfying the recurrences

$$u_n = c_{k-1}u_{n-1} + c_{k-2}u_{n-2} + \cdots + c_0 u_{n-k} + p_1(n)s_1^n,$$

and

$$v_n = c_{k-1}v_{n-1} + c_{k-2}v_{n-2} + \cdots + c_0 v_{n-k} + p_2(n)s_2^n.$$

These recurrences can be solved because the non-homogeneous part in each is in the standard form. Adding the two recurrences gives

$$u_n + v_n = c_{k-1}(u_{n-1} + v_{n-1}) + c_{k-2}(u_{n-2} + v_{n-2}) + \cdots + c_0(u_{n-k} + v_{n-k}) + p_1(n)s_1^n + p_2(n)s_2^n.$$

Therefore $a_n = u_n + v_n$ is the solution of the given recurrence.

What about the initial conditions? Suppose that the values of $a_0, a_1, a_2, \ldots, a_{k-1}$ are supplied. We need to choose $u_0, u_1, u_2, \ldots, u_{k-1}$ and $v_0, v_1, v_2, \ldots, v_{k-1}$ so that $a_n = u_n + v_n$ for $n = 0, 1, 2, \ldots, k-1$ as well. To ensure that, we can choose the initial conditions for the u and v sequences in any manner we like. For example, we can take $u_n = a_n$ and $v_n = 0$ for $n = 0, 1, 2, \ldots, k-1$.

10. Let $\{a_n\}$ be a sequence such that $a_1 = 1$, $a_{n+1} = \frac{1}{16}(1 + 4a_n + \sqrt{1 + 24a_n})$, $n \geq 1$. Find a_n.

Solution: Let us get rid of radical sign by assuming, $1+24a_n = b_n^2 \quad (\text{with } b_n > 0)$

or $a_n = \dfrac{b_n^2 - 1}{24}$, also $b_1 = 5$.

$$\Rightarrow \frac{b_{n+1}^2 - 1}{24} = \frac{1}{16}\left(1 + 4\cdot\frac{1}{24}\left(b_n^2 - 1\right) + b_n\right)$$

$$\Rightarrow 4b_{n+1}^2 - 4 = b_n^2 + 6b_n + 5$$

or
$$(2b_{n+1})^2 = (b_n + 3)^2$$

$$\Rightarrow \quad 2b_{n+1} = b_n + 3, \quad n \ge 1 \quad (\text{as } b_n > 0)$$

Let,
$$b_n = c_n + \lambda$$

$$\Rightarrow 2c_{n+1} = c_n + 3 - \lambda$$

set,
$$\lambda = 3$$

$$\Rightarrow c_{n+1} = \frac{1}{2}c_n, \, n \ge 1$$

$$\Rightarrow c_n = \left(\frac{1}{2}\right)^{n-1} c_1$$

$$\Rightarrow b_n - 3 = \left(\frac{1}{2}\right)^{n-1} (b_1 - 3)$$

$$\Rightarrow b_n = \left(\frac{1}{2}\right)^{n-1} \cdot 2 + 3 \quad (\text{as } b_1 = 5)$$

$$\Rightarrow b_n = 3 + \frac{1}{2^{n-2}}$$

$$\Rightarrow b_n^2 = 9 + \frac{1}{2^{2n-4}} + \frac{6}{2^{n-2}}$$

$$\Rightarrow a_n = \frac{1}{24}\left(8 + \frac{1}{2^{2n-4}} + \frac{6}{2^{n-2}}\right)$$

$$\text{or} \quad a_n = \frac{1 + 3\cdot 2^{n-1} + 2^{2n-1}}{3\cdot 2^{2n-1}}$$

11. Let $a_1 = 1$, $a_n = \sum_{k=1}^{n-1} (n-k)a_k$, $\forall n \ge 2$. Find a_n.

Solution: $a_1 = 1 \Rightarrow a_2 = 1, a_3 = 3, a_4 = 8$ and so on.

Then
$$a_{n+1} = \sum_{k=1}^{n} (n+1-k)a_k \qquad (1)$$

Also
$$a_n = \sum_{k=1}^{n-1} (n-k)a_k \qquad (2)$$

$$\Rightarrow \quad a_{n+1} - a_n = \sum_{k=1}^{n} a_k \quad (\text{From } (1) - (2)) \qquad (3)$$

$$\Rightarrow \quad a_{n+2} - a_{n+1} = \sum_{k=1}^{n+1} a_k \qquad (4)$$

$$\Rightarrow \quad (a_{n+2} - a_{n+1}) - (a_{n+1} - a_n) = a_{n+1} \quad (\text{From } (4) - (3))$$

$$\Rightarrow \quad a_{n+2} = 3a_{n+1} - a_n \; \forall \; n \ge 2$$

Characteristic equation is

$$x^2 - 3x + 1 = 0 \quad \Rightarrow \quad x = \frac{3 \pm \sqrt{5}}{2}$$

$$\Rightarrow \quad a_n = \lambda\left(\frac{3+\sqrt{5}}{2}\right)^n + \mu\left(\frac{3-\sqrt{5}}{2}\right)^n$$

$$a_2 = 1 = \lambda\left(\frac{3+\sqrt{5}}{2}\right) + \mu\left(\frac{3+\sqrt{5}}{2}\right)$$

$$a_3 = 3 = \lambda\left(\frac{7+3\sqrt{5}}{2}\right) + \mu\left(\frac{7-3\sqrt{5}}{2}\right)$$

$$\lambda = \frac{2}{\sqrt{5}(3+\sqrt{5})} \quad \text{and} \quad \mu = -\frac{2}{\sqrt{5}(3-\sqrt{5})}$$

$$\Rightarrow \quad a_n = \frac{(3+\sqrt{5})^{n-1} - (3-\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}} \quad \forall n \geq 2$$

12. $a_0 = 0, a_1 = 1, a_n = 2a_{n-1} + a_{n-2}, n \geq 2$. Prove that $2^k | a_n$ if and only if $2^k | n$.

Solution: By the binomial theorem, if $(1+\sqrt{2})^n = A_n + B_n\sqrt{2}$, then $(1-\sqrt{2})^n = A_n - B_n\sqrt{2}$. Multiplying these 2 equations, we get $A_n^2 - 2B_n^2 = (-1)^n$.
This implies A_n is always odd. Using characteristic equation method to solve the given recurrence relations on a_n, we find that $a_n = B_n$.

Now write $n = 2^k m$, where m is odd.

We have $k = 0$ (*i.e., n* is odd) if and only if $2B_n^2 = A_n^2 + 1 \equiv 2 \pmod 4$, (*i.e.*, B_n is odd). Next suppose case k is true.

Since $(1+\sqrt{2})^{2n} = (A_n + B_n\sqrt{2})^2 = A_{2n} + B_{2n}\sqrt{2}$, so $B_{2n} = 2A_nB_n$.

Then it follows case k implies case $k+1$.

Aliter: From given recurrence we can easily get,

$$a_n = \frac{1}{2\sqrt{2}}\left((1+\sqrt{2})^n - (1-\sqrt{2})^n\right) = \binom{n}{1} + 2\binom{n}{3} + 2^2\binom{n}{5} + \cdots$$

Let $n = 2^k$m with m being odd; then for $r > 0$ the summand

$2^r\binom{n}{2r+1} = 2^r\frac{n}{2r+1}\binom{n-1}{2r} = 2^{r+k}\frac{m}{2r+1}\binom{n-1}{2r}$ is divisible by 2^{r+k} (As $2r+1$ is odd)

Hence, $a_n = n + \sum_{r>0} 2^r\binom{n}{2r+1} = 2^k m + 2^{k+1}s$, for some integer s.

$\Rightarrow a_n$ is exactly divisible by 2^k.