CS21201 Discrete Structures
Practice Problems Solutions
Generating Functions

1. Find the generating function of the sequence $1, 2, 0, 3, 4, 0, 5, 6, 0, 7, 8, 0, \ldots$.

   Let us decompose the sequence as follows.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
		−3			−6			−9			−12	0	0	−15	...
			−1	−1		−2	−2		−3	−3		−4	−4		...
1	2	0	3	4	0	5	6	0	7	8	0	9	10	0	...

   Therefore the generating function of the given sequence is

$$\begin{aligned} & \left(1+2x+3x^2+4x^3+\cdots\right) - 3x^2\left(1+2x^3+3x^6+4x^9+\cdots\right) - (x^3+x^4)(1+2x^3+3x^6+4x^9+\cdots) \\ = & \frac{1}{(1-x)^2} - \frac{3x^2}{(1-x^3)^2} - \frac{x^3(1+x)}{(1-x^3)^2} \\ = & \frac{1+2x+x^3}{(1-x^3)^2}. \end{aligned}$$

2. Let the two-variable sequence $a_{m,n}$ be recursively defined as follows.

$$a_{m,n} = \begin{cases} 1 & \text{if } m = 0 \text{ or } n = 0, \\ a_{m-1,n} + a_{m,n-1} & \text{if } m \geqslant 1 \text{ and } n \geqslant 1. \end{cases}$$

   Find the generating function $A(x, y) = \sum_{m,n \geq 0} a_{m,n} x^m y^n$. From this, derive a closed-form formula for $a_{m,n}$.

We have

$$\begin{aligned}A(x,y) &= \sum_{m,n\geqslant 0} a_{m,n}x^m y^n \\ &= 1+\sum_{m\geqslant 1} x^m+\sum_{n\geqslant 1} y^n+\sum_{m,n\geqslant 1}(a_{m-1,n}+a_{m,n-1})x^m y^n \\ &= 1+\frac{x}{1-x}+\frac{y}{1-y}+\sum_{m,n\geqslant 1} a_{m-1,n}x^m y^n+\sum_{m,n\geqslant 1} a_{m,n-1}x^m y^n.\end{aligned}$$

Replacing $m-1$ by m in the first sum gives

$$\sum_{m,n\geqslant 1} a_{m-1,n}x^m y^n = x\sum_{\substack{m\geqslant 0\\ n\geqslant 1}} a_{m,n}x^m y^n = x\left[\sum_{\substack{m\geqslant 0\\ n\geqslant 0}} a_{m,n}x^m y^n\right]-x\left[\sum_{m\geqslant 0} x^m\right] = xA(x,y)-\frac{x}{1-x}.$$

Likewise, the second sum is $yA(x,y)-\frac{y}{1-y}$. Using these expressions gives

$$A(x,y)=\frac{1}{1-x-y}=1+(x+y)+(x+y)^2+(x+y)^3+\cdots+(x+y)^{m+n}+\cdots.$$

This gives $a_{m,n}=\binom{m+n}{m}=\binom{m+n}{n}$.

3. Prove that the generating function for Catalan numbers is $f(x)=\frac{1-\sqrt{1-4x}}{2x}$.

Solution: Start by deriving the recursive relation for Catalan numbers, given by:

$$c_0 = 1 \text{ and } c_n = \sum_{k=0}^{n-1} c_k c_{n-k-1} \text{ for every } n \geq 1$$

One way to do this is to pick the parenthesization of a product of n numbers as an example. Note that an expression with $n+1$ numbers can be split at a specific "root" multiplication, which separates the expression into two parts; a left part with k factors (C_k ways), and a right part with $n-k$ factors (C_{n-k} ways).

If $c(x)$ is the generating function of Catalan numbers, note that,

$$\begin{aligned}c(x)c(x) &= (c_0+c_1x+c_2x^2+\ldots..)(c_0+c_1x+c_2x^2+\ldots..)\\ &= c_0c_0+(c_1c_0+c_0c_1)x+(c_2c_0+c_1c_1+c_0c_2)x^2\ldots..\end{aligned}$$

In general, the coefficient of x^m in $(c(x))^2$ is,

$$\sum_{k=0}^{m} c_k c_{m-k}$$

This is the same as the coefficient of x^{m+1} in $c(x)$. Thus, we arrive at an expression for $c(x)$ in terms of $(c(x))^2$; $x(c(x))^2$ gives all of the terms of $c(x)$ except c_0. We have,

$$c(x) = x(c(x))^2 + c_0$$

or, $$x(c(x))^2 - c(x) + 1 = 0$$

Solving for $c(x)$, we have,

$$c(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Out of both roots, only $\frac{1-\sqrt{1-4x}}{2x}$ gives positive coefficients for x^n, $n \geq 0$.

4. Let a_n, $n \geq 0$, be the sequence satisfying

$$a_0 = 1$$
$$a_n = 2 + 2a_0 + 2a_1 + 2a_2 + \ldots + 2a_{n-2} + a_{n-1} \text{ for } n \geq 1$$

Deduce that the generating function of this sequence is $\frac{1+x}{1-2x-x^2}$. Solve for a_n.

The generating function of the sequence is

$$\begin{aligned} A(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \\ &= 1 + (2+a_0)x + (2+2a_0+a_1)x^2 + (2+2a_0+2a_1+a_2)x^3 + \cdots + \\ &\quad (2+2a_0+2a_1+2a_2+\cdots+2a_{n-2}+a_{n-1})x^n + \cdots \\ &= 1 + 2x\left(1+x+x^2+x^3+\cdots+x^{n-1}+\cdots\right) + 2x^2\left(a_0+(a_0+a_1)x+(a_0+a_1+a_2)x^2+\cdots\right) + \\ &\quad x\left(a_0+a_1x+a_2x^2+\cdots+a_{n-1}x^{n-1}+\cdots\right) \\ &= 1 + \frac{2x}{1-x} + 2x^2\left(\frac{A(x)}{1-x}\right) + xA(x). \end{aligned}$$

It therefore follows that

$$\left(1 - \frac{2x^2}{1-x} - x\right)A(x) = 1 + \frac{2x}{1-x},$$

that is,

$$(1-2x-x^2)A(x) = 1+x,$$

that is,

$$A(x) = \frac{1+x}{1-2x-x^2}.$$

Now, use the fact that $1-2x-x^2 = \left(1-\left(1+\sqrt{2}\right)x\right)\left(1-\left(1-\sqrt{2}\right)x\right)$.

5. How many bit strings of length n are there in which 1's always occur in contiguous pairs? You should consider strings of the form 0011011110, but not of the form 0110111110, because the last 1 is not paired.
Solution: Note that the number of such bit strings for a given length n can be defined recursively; for $n > 2$, a string either starts with 0 or with a pair 11. Using this we can say that the number of such bit strings for a given length n, $a_n = a_{n-1} + a_{n-2}$, $a_0 = a_1 = 1$. Proceed as in the case of Fibonacci numbers.

6. Let u, v, s, t be positive constant values. Consider the sequence $a_0, a_1, a_2, \ldots$. defined recursively as follows.

$$a_0 = u$$
$$a_1 = v$$
$$a_n = sa_{n-1} + ta_{n-2} \text{ for all } n \geq 2$$

For $n \geq 3$, we have

$$a_n = sa_{n-1} + ta_{n-2} = s(sa_{n-2} + ta_{n-3}) + ta_{n-2} = (s^2 + t)a_{n-2} + sta_{n-3}$$

In view of this, consider the sequence $b_0, b_1, b_2,...$ defined recursively as follows:

$$b_0 = u$$
$$b_1 = v$$
$$b_n = (s^2 + t)b_{n-2} + stb_{n-3} \text{ for all } n \geq 3$$

Demonstrate that the generating functions of both the sequences are the same.

We have

$$\begin{aligned} A(x) &= a_0 + a_1 x + \sum_{n \geqslant 2} a_n x^n \\ &= u + vx + \sum_{n \geqslant 2} (sa_{n-1} + ta_{n-2}) x^n \\ &= u + vx + sx \sum_{n \geqslant 2} a_{n-1} x^{n-1} + tx^2 \sum_{n \geqslant 2} a_{n-2} x^{n-2} \\ &= u + vx + sx\big(A(x) - u\big) + tx^2 A(x), \end{aligned}$$

that is,

$$A(x) = \frac{u + (v - su)x}{1 - sx - tx^2}.$$

On the other hand, we have

$$\begin{aligned} B(x) &= b_0 + b_1 x + b_2 x^2 + \sum_{n \geqslant 3} b_n x^n \\ &= u + vx + (sv + tu)x^2 + \sum_{n \geqslant 3} \Big((s^2 + t) b_{n-2} + st b_{n-3} \Big) x^n \\ &= u + vx + (sv + tu)x^2 + (s^2 + t)x \sum_{n \geqslant 3} b_{n-2} x^{n-2} + stx^2 \sum_{n \geqslant 3} b_{n-3} x^{n-3} \\ &= u + vx + (sv + tu)x^2 + (s^2 + t)x^2 \big(B(x) - u\big) + stx^3 B(x). \end{aligned}$$

This simplifies to

$$B(x) = \frac{u + vx + (sv + tu - s^2 u - tu)x^2}{1 - (s^2 + t)x^2 - stx^3} = \frac{u + vx + (sv - s^2 u)x^2}{1 - (s^2 + t)x^2 - stx^3}.$$

Now, look at the numerator and the denominator of $A(x)$. We have

$$(1 + sx)(u + (v - su)x) = u + (v - su)x + sux + s(v - su)x^2 = u + vx + (sv - s^2 u)x^2,$$

whereas

$$(1 + sx)(1 - sx - tx^2) = 1 - sx - tx^2 + sx - s^2 x^2 - stx^3 = 1 - (s^2 + t)x^2 - stx^3.$$

We can therefore cancel $1 + sx$ from the numerator and the denominator of $B(x)$ to conclude that $A(x) = B(x)$.

7. Let $a_n = \sum_{i=n}^{\infty} \frac{2^i}{i!}$ for all integers $n \geq 0$.

(a) Find a closed-form expression for the (ordinary) generating function $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n + ...$ of the sequence $a_0, a_1, a_2, ... a_n, ...$

Solution:

$$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n + ...$$
$$= (\sum_{i=0}^{\infty} \frac{2^i}{i!}) + (\sum_{i=1}^{\infty} \frac{2^i}{i!})x + (\sum_{i=2}^{\infty} \frac{2^i}{i!})x^2 + ... + (\sum_{i=0}^{\infty} \frac{2^i}{i!})x^n + ...$$
$$= \sum_{n=0}^{\infty} (\sum_{i=n}^{\infty} \frac{2^i}{i!})x^n$$
$$= \sum_{n=0}^{\infty} [(1 + x + x^2 + x^3 + ... + x^n)\frac{2^n}{n!}]$$
$$= \sum_{n=0}^{\infty} [(\frac{1-x^{n+1}}{1-x})\frac{2^n}{n!}]$$
$$= \frac{1}{1-x} [\sum_{n=0}^{\infty} (\frac{2^n}{n!}) - x(\sum_{n=0}^{\infty} \frac{(2x)^n}{n!})]$$
$$= \frac{e^x - xe^2}{1-x}$$

(b) Use the expression for $A(x)$ in Part (a) to prove that $\sum_{n=0}^{\infty} a_n = 3e^2$.

Solution: The desired sum is $A(1)$. But $A(1)$ is of the form 0/0, so the desired sum is $\lim_{x\to1} A(x)$, provided the limit exists. By using l'Hôpital's rule, we get:

$$\sum_{n=0}^{\infty} a_n = \lim_{x\to1} A(x) = \lim_{x\to1} \frac{e^2 - xe^{2x}}{1-x} = \lim_{x\to1} \frac{-(e^{2x} + 2xe^{2x})}{-1} = 3e^2$$

CS21201 Discrete Structures
Tutorial Solutions
Generating Functions

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1. Let F_n, $n \geq 0$, denote the Fibonacci numbers. Prove that $\sum_{n\in\mathbb{N}_0} \frac{F_n}{2^n} = 2$.

   Solution: Derive the general expression for the generating function of the Fibonacci sequence. The proof below can then be followed:

   The generating function of the Fibonacci sequence is

   $$\sum_{n\in\mathbb{N}_0} F_n x^n = \frac{x}{1-x-x^2} = \frac{x}{(1-\rho x)(1-\bar{\rho}x)} = \frac{A}{1-\rho x} + \frac{B}{1-\bar{\rho}x},$$

   where $\rho = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$ is the golden ratio, $\bar{\rho} = \frac{1-\sqrt{5}}{2} - 0.6180339887\cdots$ is the conjugate of the golden ratio, and A, B are constant real numbers. Since $|\rho/2|$ and $|\bar{\rho}/2|$ are less than 1, the power series converge for $x = \frac{1}{2}$, and we get

   $$\sum_{n\in\mathbb{N}_0} \frac{F_n}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}-(\frac{1}{2})^2} = 2.$$

2. Let l_n be the number of lines printed by the call $f(n)$ for some integer $n \geq 0$.

```
void f ( int n )
{
    int i, j;
    printf("Hi\n");
    if (n == 0) return;
    for (i=0; i<=n-1; ++i)
        for (j=0; j<=i; ++j)
            f(j);
}
```

   (a) Let $L(x) = l_0 + l_1 x + l_2 x^2 + \ldots.\ l_n x^n + \ldots.$ be the generating function of the sequence $l_0, l_1, l_2, \ldots.$ Prove that $L(x) = \frac{1-x}{1-3x+x^2}$.

   (b) Derive an explicit formula for l_n (valid for all $n \geq 0$) from the generating function $L(x)$.

The function gives the following recurrence relation for the sequence.

$$\begin{aligned} l_0 &= 1, \\ l_n &= 1 + l_{n-1} + 2l_{n-2} + 3l_{n-3} + \cdots + nl_0 \quad \text{for } n \geqslant 1. \end{aligned}$$

Therefore we have

$$\begin{aligned} L(x) &= l_0 + \sum_{n\geqslant 1} l_n x^n \\ &= 1 + \sum_{n\geqslant 1} (1 + l_{n-1} + 2l_{n-2} + 3l_{n-3} + \cdots + nl_0)x^n \\ &= \sum_{n\geqslant 0} x^n + \sum_{n\geqslant 1} (l_{n-1} + 2l_{n-2} + 3l_{n-3} + \cdots + nl_0)x^n \\ &= \frac{1}{1-x} + x\sum_{n\geqslant 0} (l_n + 2l_{n-1} + 3l_{n-2} + \cdots + (n+1)l_0)x^n \\ &= \frac{1}{1-x} + \frac{xL(x)}{(1-x)^2}. \end{aligned}$$

It follows that

$$(1-x)^2 L(x) = 1 - x + xL(x),$$

that is,

$$L(x) = \frac{1-x}{1-3x+x^2}.$$

We have

$$\begin{aligned} L(x) &= \frac{1-x}{1-3x+x^2} \\ &= \frac{1-x}{\left(1-\left(\frac{3+\sqrt{5}}{2}\right)x\right)\left(1-\left(\frac{3-\sqrt{5}}{2}\right)x\right)} \\ &= \frac{A}{1-\left(\frac{3+\sqrt{5}}{2}\right)x} + \frac{B}{1-\left(\frac{3-\sqrt{5}}{2}\right)x}. \end{aligned}$$

Solving for A and B (show the steps) gives

$$A = \frac{\sqrt{5}+1}{2\sqrt{5}}, \qquad B = \frac{\sqrt{5}-1}{2\sqrt{5}}.$$

Therefore

$$l_n = \left(\frac{\sqrt{5}+1}{2\sqrt{5}}\right)\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{3-\sqrt{5}}{2}\right)^n \quad \text{for all } n \geqslant 0.$$

3. Let $a_0, a_1, a_2, \ldots, a_n, \ldots$ be the sequence generated by $\sum_{r\in\mathbb{N}} \frac{x^r}{1-x^r}$. Denote by p_n the parity of a_n, that is, $p_n = 0$ if a_n is even and $p_n = 1$ if a_n is odd. Determine all n ∈ N, for which $p_n = 1$. Justify.

Solution: The sequence is such that a_n is the number of divisors of n. This can be shown by expanding the given relation. For $r \in \mathbb{N}$,

$$\frac{x^r}{1-x^r} = x^r + x^{2r} + x^{3r} + ...$$

$$\sum_{r\in\mathbb{N}} \frac{x^r}{1-x^r} = \sum_{r\in\mathbb{N}} \sum_{k=1}^{\infty} x^{kr}$$

The double sum represents the number of ways each power of x can be formed from the sums of terms kr. The number of times a particular x^n appears in this sum is the number of divisors of n. More precisely, the coefficient of x^n in the series corresponds to the number of divisors of n. Say, $a_n = d(n),\ n \in \mathbb{N}$.

The only time $d(n)$ is odd is when n is a perfect square (since $n = k^2,\ k \in \mathbb{N}$ implies every divisor x of k is paired with n/x to give 2 divisors for n, except k itself). Therefore, $p_n = 1$ holds for all n that are perfect squares.
