

## Tutorial Solution: Set Sizes

1.

First, we claim that the set of all polynomials with rational coefficients is countable ( $P_n$ )

Proof: -  $P_n = \{C_0 + C_1x + C_2x^2 + \dots + C_nx^n : C_n \neq 0, C_0, C_1, \dots, C_{n-1} \in \mathbb{Q}\}$   
 $|P_n| = |\mathbb{Q} \setminus \{0\} \times \mathbb{Q}^{n-1}|$  Cartesian Product of a finite number of countable sets  $\mathbb{Q} \setminus \{0\} \times \mathbb{Q}^{n-1}$  is countable.  $\Rightarrow P_n$  is countable

The set of all polynomials with rational coefficients  
 $P = \bigcup_{n=1}^{\infty} P_n$

[Countable as it is a countable union of countable sets]

Each polynomial has a finite number of roots.  $\Downarrow$

Let  $A$  be the set of all algebraic numbers.  $\Rightarrow A$  is countable

Transcendental numbers  $= \mathbb{R} \setminus A$  is uncountable as per the previous proof.

**Clarification of the last part:** If  $A \subset \mathbb{R}$  is countable, then  $\mathbb{R} \setminus A$  is uncountable. Otherwise,  $A \cup \mathbb{R} \setminus A = \mathbb{R}$  would have been countable (union of countable sets is countable).

Let  $I_n$  denote the set of sequences that differ from  $A$  only in the first  $n$  digits.

$\Rightarrow I_n$  is a finite set with at most  $2^n$  elements

We list  $I_1, I_2, I_3, \dots \Rightarrow$  list of all binary sequences equivalent to  $A$

each of these sets are countable

$\&$   $\bigcup_{n=1}^{\infty} I_n$  is a countable union of countable sets

$\rightarrow$  There are countably many binary sequences equivalent to  $A$

2.

Consider  $A$  and  $B$  such that  $|A| = |B| = |\mathbb{R}|$ . It is obvious that  $|A| = |\mathbb{R}| \leq |A \cup B|$ . We therefore prove that  $|A \cup B| \leq |\mathbb{R}|$ . We provide an injective map  $f : A \cup B \rightarrow [0, 2]$ . It is already known that  $|\mathbb{R}| = |[0, 1]| = |[0, 2]|$ .

Let  $g_1 : A \rightarrow [0, 1]$  and  $g_2 : B \rightarrow [1, 2]$ .

Let  $A = \{a_1, a_2, a_3, a_4, \dots\}$   $B = \{b_1, b_2, b_3, b_4, \dots\}$ . and  $A \cup B = \{a_1, b_1, a_2, a_3, b_3, \dots\}$

Define the injective map  $f : A \cup B \rightarrow [0, 2]$  as:

$$f(x) = \begin{cases} g_1(x), & x \in A \\ g_2(x), & x \notin A \end{cases}$$

Is this map bijective?

- Note: We cannot assume direct linear mapping from sets  $A, B$  to  $\mathbb{R}$ , since  $A, B$  may not be sets of integers, but sets of anything, like strings, other sets, etc.

3.

a) Prove that there is a map  $A \rightarrow A$  which is injective but not surjective.

Pick any countable subset  $B = \{b_1, b_2, b_3, \dots\}$  of  $A$ . Define the map  $f : A \rightarrow A$  as

$$f(a) = \begin{cases} b_{n+1} & \text{if } a = b_n \\ a & \text{otherwise} \end{cases}$$

b) Prove that there is a map  $A \rightarrow A$  which is surjective but not injective.

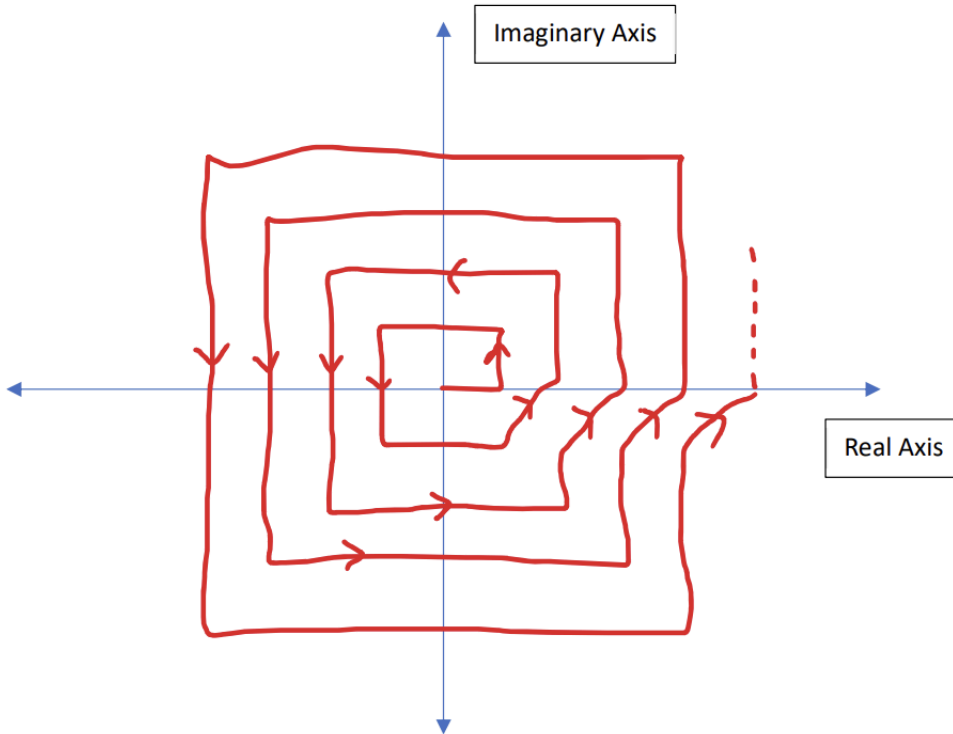
Let  $B$  be the same as in (a). Pick the map  $g : A \rightarrow A$  as

$$f(a) = \begin{cases} b_1 & \text{if } a = b_1 \\ b_{n-1} & \text{if } a = b_n \text{ for } n \geq 2 \\ a & \text{otherwise} \end{cases}$$

## Practice Solution: Set Sizes

1.

The bijection is given by the exhaustive enumeration shown below in terms of the coordinate system, note that the same element is not visited again.



An alternate way could be to see that  $|A| = |\mathbb{Z} \times \mathbb{Z}|$ . Then we could prove that  $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{Z}|$ . Without loss of generality, we need to prove that the cartesian product of two countable sets  $A$  and  $B$  is countable.

Proof:  $\forall a \in A$ , define  $B_a = \{(a, b) \mid b \in B\}$ . Clearly  $f: B \rightarrow B_a$  is a bijection as we can map each instance  $b \in B$  to  $(a, b) \in B_a$ . Hence each  $B_a$  is countable.

By definition:

$$A \times B = \bigcup_{a \in A} B_a$$

Since the union of countably many countable sets is countable,  $A \times B$  is countable.

2.

Prove that the set of all permutations of  $\mathbb{N}$  is not countable.

The set of all permutations of  $\mathbb{N}$  is equinumerous with the set of all bijective functions  $f: \mathbb{N} \rightarrow \mathbb{N}$ . Notice that each permutation is basically a function mapping a place (which is a natural number) to a natural number.

It is easy to see using a diagonalization argument that the set of all bijective functions  $f: \mathbb{N} \rightarrow \mathbb{N}$  is uncountable.

3.

Consider the set  $S = \{a + b\sqrt{7} \mid a, b \in \mathbb{Z}\}$ . Prove that  $\mathbb{R} - S$  is uncountable.

Note that  $S$  is countable, since  $f : \mathbb{Z} \times \mathbb{Z} \rightarrow S$  is a bijection. Assume that  $\mathbb{R} - S$  is countable, then  $(\mathbb{R} - S) \cup S = \mathbb{R}$  is countable, since it will be the union of two countable sets. However,  $\mathbb{R}$  is uncountable and therefore our assumption is wrong.

By the same logic, for any countable set  $S$ ,  $\mathbb{R} - S$  is uncountable.

4.

Provide an explicit bijection between  $\mathbb{N}$  and  $\mathbb{N} \times \mathbb{N}$ . It should not be an exhaustive enumeration.

Think about the exhaustive enumeration method where you tried to enumerate all the coordinates in a diagonal scheme.

For a natural number  $N$ , try to find out how many diagonals can be completely enumerated.

First diagonal enumerates (1, 1) : 1 point  
Second diagonal enumerates (2, 1) and (1, 2) : 2 points

Third diagonal enumerates (3, 1), (2, 2) and (1, 3) : 3 points  
 $n$ th diagonal enumerates  $(n, 1), (n - 1, 2) \dots (1, n)$  :  $n$  points

This is the maximum  $n$  such that  $\frac{n(n+1)}{2} \leq N$ . Find this  $n$ . Let  $d = N - n$

[ $d = 0$ ] The mapping is given by  $N \rightarrow (1, n)$

[ $0 < d \leq n + 1$ ] The mapping is given by  $N \rightarrow (n + 1 - (d - 1), 1 + (d - 1)) = (n - d + 2, d)$

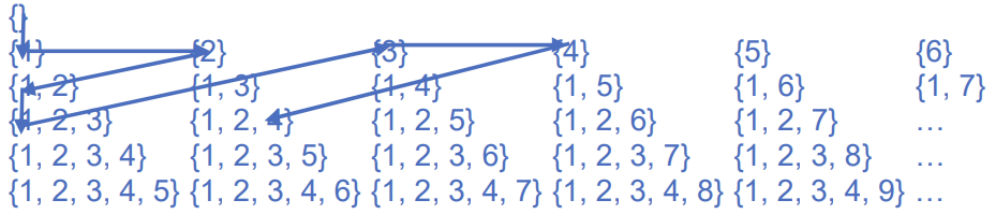
5.

Determine whether the following sets are countable or uncountable:

a) The set of all finite subsets of  $\mathbb{N}$

Countable

Let this set be  $A$ . We find a way to enumerate all the finite subsets of  $\mathbb{N}$  using the following enumeration scheme



Alternate Method: We also know that a countable union of countable sets is countable. The above set can be written as  $S = S_0 \cup S_1 \cup S_2 \cup S_3 \cup \dots \cup S_n \cup \dots$  where  $S_i \subseteq \mathbb{N}^i$ . Each of  $\mathbb{N}^i$  is countable.

b) The set of all infinite subsets of  $\mathbb{N}$

Uncountable

Let this set be  $B$ . Assume that  $B$  is countable, then  $A \cup B$  is also countable, since it is a union of a countable number of countable sets. But  $A \cup B$  is the number of subsets of  $\mathbb{N}$  which is obviously uncountable. Therefore, our assumption is wrong and  $B$  is uncountable.

6.

Infinite Bit Sequences As the name suggests, an infinite bit sequence is an infinite sequence of 0s and 1s. Denote  $S$  as the set of all infinite bit sequences. Let  $\alpha(n)$  be the  $n$ th element of an infinite bit sequence  $\alpha \in S$ . Determine whether the following sets are countable or uncountable:

a)  $S$

Uncountable

A simple diagonalization argument would suffice. Construct an infinite bit sequence  $\beta$  such that  $\beta(n) = \overline{\alpha_n(n)}$ , where  $\alpha_n$  denotes the  $\alpha$  which is mapped to integer  $n \in \mathbb{N}$ . Clearly,  $\beta \in S$ . Let  $\beta = \alpha_k$  for some  $k \in \mathbb{N}$ . But  $\beta(k) \neq \alpha_k(k)$  by virtue of the above construction. Hence  $S$  is uncountable

b)  $T_1 = \{\alpha \in S \mid \alpha(n) = 1 \text{ and } \alpha(n+1) = 0 \text{ for some } n \geq 0\}$

Uncountable

Consider the set  $T_3 = \{\alpha \in S \mid \alpha(0) = 1 \text{ and } \alpha(1) = 0\}$ . We have  $T_3 \subseteq T_1$ , and so  $|T_3| \leq |T_1| \leq |S|$  (use the canonical inclusion maps which are injective).

On the other hand, take any  $\alpha = (1, 0, a_2, a_3, \dots, a_n, \dots) \in T_3$ . The map taking  $\alpha \rightarrow (a_2, a_3, a_4, \dots, a_{n+2}, \dots) \in S$  is clearly a bijection  $T_3 \rightarrow S$ , implying that  $|T_3| = |S|$ .

c)  $T_2 = \{\alpha \in S \mid \alpha(n) = 1 \text{ and } \alpha(n+1) = 0 \text{ for no } n \geq 0\}$

Countable

Each sequence of  $T_2$  is of the format 0000...00001111... or 0000...

Consider the bijective map  $f : T_2 \rightarrow \mathbb{N}$  such that  $f(0000 \dots) = 1$

and for all other sequences  $\alpha \in T_2, f(\alpha) = n + 2$ , where  $n$  is the number of zeros.

7.

**[Sets of Functions]** We have the following sets, determine whether they are countable or uncountable.

a) The set of all functions from  $\mathbb{N}$  to  $\{1, 2\}$

Uncountable

Assume that this set ( $S$ ) is countable. Therefore, there is a bijective mapping  $f : \mathbb{N} \rightarrow S$  such that  $a \in \mathbb{N} \rightarrow f_a \in S$ . Consider the following diagonalization argument:

	$f_1$	$f_2$	$f_3$	$f_4$	..	..	$f_n$	..	..
1	1	2	2	1	..	..	2	..	..
2	1	1	2	1	..	..	1	..	..
3	2	2	1	2	..	..	2	..	..
4	1	1	1	2	..	..	1	..	..
..	..	..	..	..	..	..	..	..	..
..	..	..	..	..	..	..	..	..	..
n	2	1	2	2	..	..	1	..	..
..	..	..	..	..	..	..	..	..	..

Consider a function  $F$  which gives outputs opposite of those given by  $f_i$  across the diagonal above.

$$F(x) = \begin{cases} 1, & f_x(x) = 2 \\ 2, & f_x(x) = 1 \end{cases} \quad \forall x \in \mathbb{N}$$

$F \in S$ . However, this means that  $F = f_k$  for some  $k$ . But, if  $f_k(k) = 1$ , then  $F(k) = 2$ . Hence our assumption is invalid.

b) The set of all functions from  $\mathbb{N}$  to  $\mathbb{N}$

Uncountable, any superset of an uncountable set is also uncountable. Alternatively, the above diagonalization argument can be modified to prove this statement by selecting  $F(x) = f_x(x) + 1$ .

c) The set of all functions from  $\{1, 2\}$  to  $\mathbb{N}$

Countable, notice that this set will have the same cardinality as  $\mathbb{N} \times \mathbb{N}$ . Let this set be  $S$ . Each  $f \in S$  has two values  $f(1)$  and  $f(2)$ , both of which are in  $\mathbb{N}$ .

d) The set of all non-increasing functions from  $\mathbb{N}$  to  $\mathbb{N}$

Countable, this set is the countable union of countable sets. Let this set be denoted by  $S$ . Let  $f \in S$ . Then:

$$f(1) \geq f(2) \geq f(3) \geq \dots$$

Eventually,  $\forall f \in S$ , there exist values  $i$  and  $n_0$  such that  $\forall n \geq n_0 [f(n) = i]$ .

Mathematically,

$$\exists n_0 \exists i (\forall n \geq n_0) [f(n) = i]$$

For a fixed  $(n_0, i)$  pair, let  $F_{\{n_0, i\}}$  be the set of non-increasing functions such that  $(\forall n \geq n_0) [f(n) = i]$ .

Let  $g \in F_{\{n_0, i\}}$ . We map  $g$  to the ordered  $(i - 1)$ -tuple

$$\{g(1), g(2), \dots, g(i - 1)\}$$

Notice that this map is a bijection subject to the co-domain satisfying the conditions

$g(j) \geq g(j + 1)$  and  $g(i - 1) \geq i$ .

The cardinality of the set  $[g(1), g(2), \dots, g(i - 1)]$  is bounded by  $|\mathbb{R}^{i-1}|$  and therefore by  $|\mathbb{R}|$ .

By the Cantor-Schroder-Bernstein theorem,  $F_{\{n_0, i\}}$  is countable. Hence

$$S = \bigcup_{n_0} \bigcup_i F_{\{n_0, i\}}$$

is also countable.