Discrete Structures 2024

Sets, Relation, Function - Practice Problems

September 2024

1. Let A, B, $C \subseteq U$. Prove that $(A - B) \subseteq C$ if and only if $(A - C) \subseteq B$.

   Solution:

   We need to prove the statement:

   $$(A - B) \subseteq C \iff (A - C) \subseteq B$$

   This is an "if and only if" statement, so we will prove both directions:

   (a). $(A - B) \subseteq C \implies (A - C) \subseteq B$

   Assume that $(A - B) \subseteq C$. We want to show that $(A - C) \subseteq B$.

   Let $x \in (A - C)$. By the definition of set difference:

   $$x \in A \quad \text{and} \quad x \notin C$$

   Since $x \in A$ and $x \notin C$, it remains to show that $x \in B$.

   Assume for contradiction that $x \notin B$. This implies $x \in A$ and $x \notin B$, so $x \in (A - B)$. But we are given that $(A - B) \subseteq C$, so $x \in C$, which contradicts $x \notin C$.

   Thus, $x \in B$, and we conclude that $(A - C) \subseteq B$.

   (b). $(A - C) \subseteq B \implies (A - B) \subseteq C$

   Now assume that $(A - C) \subseteq B$. We want to show that $(A - B) \subseteq C$.

   Let $x \in (A - B)$. By the definition of set difference:

   $$x \in A \quad \text{and} \quad x \notin B$$

   Since $x \in A$ and $x \notin B$, it remains to show that $x \in C$.

   Assume for contradiction that $x \notin C$. This implies $x \in A$ and $x \notin C$, so $x \in (A - C)$. But we are given that $(A - C) \subseteq B$, so $x \in B$, which contradicts $x \notin B$.

   Thus, $x \in C$, and we conclude that $(A - B) \subseteq C$.

   Conclusion:

   Since both directions have been proven, we have shown that:

   $$(A - B) \subseteq C \iff (A - C) \subseteq B$$

2. Let $A, B \subseteq \mathbb{R}$, where

   $$A = \{x \mid x^2 - 7x \leq -12\} \quad \text{and} \quad B = \{x \mid x^2 - x \leq 6\}.$$

   Determine $A \cup B$ and $A \cap B$.

   Solution:

   $$x^2 - 7x \leq -12 \implies x^2 - 7x + 12 \leq 0 \implies (x - 3)(x - 4) \leq 0$$

   $$[(x - 3) \leq 0 \text{ and } (x - 4) \geq 0] \text{ or } [(x - 3) \geq 0 \text{ and } (x - 4) \leq 0]$$

$$\implies [x \le 3 \text{ and } x \ge 4] \text{ or } [x \ge 3 \text{ and } x \le 4] \implies 3 \le x \le 4,$$

so $A = \{x \mid 3 \le x \le 4\} = [3, 4]$.

$$x^2 - x \le 6 \implies x^2 - x - 6 \le 0 \implies (x-3)(x+2) \le 0$$
$$[(x-3) \le 0 \text{ and } (x+2) \ge 0] \text{ or } [(x-3) \ge 0 \text{ and } (x+2) \le 0]$$
$$\implies [x \le 3 \text{ and } x \ge -2] \text{ or } [x \ge 3 \text{ and } x \le -2] \implies -2 \le x \le 3,$$

so $B = \{x \mid -2 \le x \le 3\} = [-2, 3]$.

Consequently,

$$A \cap B = \{3\} \quad \text{and} \quad A \cup B = [-2, 4].$$

3. Define a relation ρ on $\mathbb{N}$ as $a \,\rho\, b$ if and only if a has the same set of prime divisors as b. For example, 5 is related to $25 = 5^2$, $12 = 2^2 \times 3$ is related to $54 = 2 \times 3^3$, but 12 is not related to $16 = 2^4$, nor to $180 = 2^2 \times 3^2 \times 5$.

   (a) Prove that ρ is an equivalence relation on $\mathbb{N}$.

   (b) Find a unique representative from each equivalence class of ρ.

Solution:

(a) Proving that ρ is an equivalence relation on $\mathbb{N}$.

To show that ρ is an equivalence relation, we need to check the following three properties: reflexivity, symmetry, and transitivity.

1. Reflexivity: We need to show that for all $a \in \mathbb{N}$, $a \,\rho\, a$. This means that a has the same set of prime divisors as itself. Since the prime divisors of a number are the same as its own prime divisors, reflexivity holds.

2. Symmetry: We need to show that if $a \,\rho\, b$, then $b \,\rho\, a$. By definition, $a \,\rho\, b$ means that a and b have the same set of prime divisors. Clearly, if a and b have the same set of prime divisors, then b and a also have the same set of prime divisors, so symmetry holds.

3. Transitivity: We need to show that if $a \,\rho\, b$ and $b \,\rho\, c$, then $a \,\rho\, c$. If $a \,\rho\, b$, this means a and b have the same set of prime divisors. Similarly, if $b \,\rho\, c$, then b and c have the same set of prime divisors. Therefore, a and c must have the same set of prime divisors, which implies $a \,\rho\, c$. Hence, transitivity holds.

Since reflexivity, symmetry, and transitivity all hold, ρ is an equivalence relation on $\mathbb{N}$.

(b) Finding a unique representative from each equivalence class of ρ.

A non-zero integer is called *square-free* if it is not divisible by the square of a prime number. Each equivalence class of ρ contains a unique square-free integer, and these unique square-free integers are different in distinct equivalence classes. To see why, let $a \in \mathbb{N}$ have the prime factorization $a = p_1^{e_1} \cdot \ldots \cdot p_t^{e_t}$ with $t > 0$, with pairwise distinct primes $p_1, \ldots, p_t$, and with each $e_i > 0$. But then $[a] = [p_1 \cdot \ldots \cdot p_t]$. Moreover, two different square-free integers have different sets of prime divisors. So we can take square-free integers as the representatives of the equivalence classes.

4. Let $f : \mathbb{Z} - \{0\} \to \mathbb{N}$ be defined by

$$f(x) = 2x - 1 \quad \text{if } x > 0, \quad \text{and} \quad f(x) = -2x \quad \text{for } x < 0.$$

   (a) Prove that f is one-to-one and onto.

   (b) Determine f^{-1}.

Solution:

(a) To prove that f is one-to-one and onto:

One-to-one (Injective): Suppose that $x_1, x_2 \in \mathbb{Z}$ and $f(x_1) = f(x_2)$. Then either $f(x_1), f(x_2)$ are both even or they are both odd.

If they are both even, then $f(x_1) = f(x_2) \Rightarrow -2x_1 = -2x_2 \Rightarrow x_1 = x_2$.

Otherwise, $f(x_1), f(x_2)$ are both odd and $f(x_1) = f(x_2) \Rightarrow 2x_1 - 1 = 2x_2 - 1 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$.

Consequently, the function f is one-to-one.

Onto (Surjective): In order to prove that f is an onto function, let $n \in \mathbb{N}$.

If n is even, then $(-n/2) \in \mathbb{Z}$ and $(-n/2) < 0$, and $f(-n/2) = -2(-n/2) = n$.

For the case where n is odd, we find that $(n+1)/2 \in \mathbb{Z}$ and $(n+1)/2 > 0$, and $f((n+1)/2) = 2[(n+1)/2] - 1 = (n+1) - 1 = n$.

Hence f is onto.

(b) The determination of f^{-1} is left as an exercise.

5. In ten days, Ms. Rosatone typed 84 letters to different clients. She typed 12 of these letters on the first day, seven on the second day, and three on the ninth day, and she finished the last eight on the tenth day. Show that for three consecutive days, Ms. Rosatone typed at least 25 letters.

Solution:

For $1 \leq i \leq 10$, let x_i be the number of letters typed on day i. Then $x_1 + x_2 + x_3 + \ldots + x_8 + x_9 + x_{10} = 84$, or $x_3 + \ldots + x_8 = 54$. Suppose that $x_1 + x_2 + x_3 < 25$, $x_2 + x_3 + x_4 < 25, \ldots, x_8 + x_9 + x_{10} < 25$. Then $x_1 + 2x_2 + 3(x_3 + \ldots + x_8) + 2x_9 + x_{10} < 8(25) = 200$, or $3(x_3 + \ldots + x_8) < 160$. Consequently, $54 = x_3 + \ldots + x_8 < (160)/3 = 53\frac{1}{3}$.

6. Suppose you set your computer password of length m from a fixed chosen set of n different characters available in the keyboard ($m \geq n$). How many different passwords can you set so that at least one occurrence of each symbol (from the n chosen set of keyboard symbols) will be present?

Solution:

1. We want to find the number of passwords that:
   - Have length m
   - Use n different characters
   - Include at least one of each of the n characters
2. This is an application of the Inclusion-Exclusion Principle.
3. The total number of possible passwords of length m using n characters is n^m.
4. We need to subtract the number of passwords that are missing at least one character.
5. Let A_i be the set of passwords missing the i-th character. We want to find $|A_1 \cup A_2 \cup \ldots \cup A_n|$.
6. Thus,

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

7. Calculating each term:
   - $|A_i| = (n-1)^m$ (passwords using only $n-1$ characters)
   - $|A_i \cap A_j| = (n-2)^m$ (passwords using only $n-2$ characters)
   - $\vdots$
   - $|A_1 \cap A_2 \cap \ldots \cap A_n| = 0$ (no password can miss all characters)
8. The number of valid passwords is:

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \ldots + (-1)^{n-1}\binom{n}{n-1}(1)^m$$

Therefore, the number of different passwords of length m using n characters where each character appears at least once is:

$$n^m - \sum_{k=1}^{n-1} (-1)^{k-1} \binom{n}{k} (n-k)^m$$

7. For $m, n \in \mathbb{Z}^+$ with $m < n$, prove that,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m = 0.$$

Solution: Application of Inclusion-Exclusion Principle

8. Let $A = \mathbb{R} \times \mathbb{R}$ be the set of all ordered pairs of real numbers. Define a binary relation $\diamond$ on A as follows:

For $(a_1, b_1), (a_2, b_2) \in A$,

$$(a_1, b_1) \diamond (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + b_1 a_2)$$

(a) Prove or disprove that $\diamond$ is commutative.

(b) Prove or disprove that $\diamond$ is associative.

(c) Find the identity element for $\diamond$, if it exists.

(d) For any $(a, b) \in A$ where $a^2 + b^2 \neq 0$, find the inverse element under $\diamond$.

Solution:

(a) Commutativity:
Let $(a_1, b_1), (a_2, b_2) \in A$. We need to show that $(a_1, b_1) \diamond (a_2, b_2) = (a_2, b_2) \diamond (a_1, b_1)$.

$$(a_1, b_1) \diamond (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + b_1 a_2)$$
$$(a_2, b_2) \diamond (a_1, b_1) = (a_2 a_1 - b_2 b_1, a_2 b_1 + b_2 a_1)$$

Since multiplication of real numbers is commutative, these are equal. Thus, $\diamond$ is commutative.

(b) Associativity:
Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in A$. We need to show that $((a_1, b_1) \diamond (a_2, b_2)) \diamond (a_3, b_3) = (a_1, b_1) \diamond ((a_2, b_2) \diamond (a_3, b_3))$.
Left side:

$$\begin{aligned}
&((a_1, b_1) \diamond (a_2, b_2)) \diamond (a_3, b_3) \\
&= (a_1 a_2 - b_1 b_2, a_1 b_2 + b_1 a_2) \diamond (a_3, b_3) \\
&= ((a_1 a_2 - b_1 b_2) a_3 - (a_1 b_2 + b_1 a_2) b_3, (a_1 a_2 - b_1 b_2) b_3 + (a_1 b_2 + b_1 a_2) a_3) \\
&= (a_1 a_2 a_3 - b_1 b_2 a_3 - a_1 b_2 b_3 - b_1 a_2 b_3, a_1 a_2 b_3 - b_1 b_2 b_3 + a_1 b_2 a_3 + b_1 a_2 a_3)
\end{aligned}$$

Right side:

$$\begin{aligned}
&(a_1, b_1) \diamond ((a_2, b_2) \diamond (a_3, b_3)) \\
&= (a_1, b_1) \diamond (a_2 a_3 - b_2 b_3, a_2 b_3 + b_2 a_3) \\
&= (a_1 (a_2 a_3 - b_2 b_3) - b_1 (a_2 b_3 + b_2 a_3), a_1 (a_2 b_3 + b_2 a_3) + b_1 (a_2 a_3 - b_2 b_3)) \\
&= (a_1 a_2 a_3 - a_1 b_2 b_3 - b_1 a_2 b_3 - b_1 b_2 a_3, a_1 a_2 b_3 + a_1 b_2 a_3 + b_1 a_2 a_3 - b_1 b_2 b_3)
\end{aligned}$$

These are equal, so $\diamond$ is associative.

(c) Identity element:
Let (e, f) be the identity element. Then for any $(a, b) \in A$:

$$(a, b) = (a, b) \diamond (e, f) = (ae - bf, af + be)$$

This implies $ae - bf = a$ and $af + be = b$ for all a, b. The only solution is $e = 1$ and $f = 0$. Therefore, the identity element is $(1, 0)$.

(d) Inverse element:
For $(a, b) \in A$ where $a^2 + b^2 \neq 0$, let (x, y) be its inverse. Then:

$$(a, b) \diamond (x, y) = (1, 0)$$

This gives us:

$$ax - by = 1$$
$$ay + bx = 0$$

Solving this system:

$$x = \frac{a}{a^2 + b^2}, \quad y = \frac{-b}{a^2 + b^2}$$

Therefore, the inverse of (a, b) is $(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$.

9. Let $S = \mathbb{Z}^+ \times \mathbb{Z}^+$. Define a relation R_1 on S as, $(x, y)R_1(m, n)$ if and only if $x \leq m$ and $y \leq n$. Prove or disprove that R_1 is a total-order on S.

Solution: Given an counter example and explain. For example, (6,7),(7,6) does not belong to R1.

10. A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken a course in at least one of Spanish French and Russian, how many students have taken a course in all 3 languages.

Solution: Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian. Then, we have $|S| = 1232$, $|F| = 879$, $|R| = 114$, $|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$, and $|S \cup F \cup R| = 2092$.

Using the equation:

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$$

we obtain:

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$$

Solving for $|S \cap F \cap R|$ yields:

$$|S \cap F \cap R| = 7$$

11. Professor Bailey has just completed writing the final examination for his course in advanced engineering mathematics. This examination has 12 questions, whose total value is to be 200 points. In how many ways can Professor Bailey assign the 200 points if (a) each question must count for at least 10, but no more than 25, points? (b) each question must count for at least 10, but not more than 25, points and the point value for each question is to be a multiple of 5?

(a) Each question must count for at least 10, but no more than 25 points:

1) First, we need to allocate the minimum 10 points to each question: $12 \times 10 = 120$ points

2) Now we have 200 - 120 = 80 points left to distribute among the 12 questions.

3) Each question can receive up to 15 additional points (because 25 - 10 = 15).

4) This is equivalent to distributing 80 indistinguishable objects (points) into 12 distinguishable boxes (questions), where each box can hold up to 15 objects.

$$\sum_{k=0}^{12}(-1)^k\binom{12}{k}\binom{80+12-1-k(15+1)}{12-1}$$

(b) Each question must count for at least 10, but not more than 25, points and the point value for each question is to be a multiple of 5:

Similar to part (a), but now we're partitioning 40 (because 200/5 = 40) into 12 parts After subtracting the minimum (2 for each part, because 10/5 = 2), we're partitioning 16 into 12 parts Each part can now be from 0 to 3 (because (25-10)/5 = 3) 3)So, our new problem is: distribute 16 indistinguishable objects (each representing 5 points) into 12 distinguishable boxes, where each box can contain 0 to 3 objects.
We want to find the number of integer solutions to: $\sum_{i=1}^{12} x_i = 16, \text{ where } 0 \leq x_i \leq 3 \text{ for all } i \in \{1, 2, \ldots, 12\}$

Discrete Structures 2024

Sets, Relation, Function - Tutorial Problems

September 12th 2024

1. $S = \{(1,2),(2,1)\}$ is a binary relation on the set $A = \{1,2,3\}$. Is it irreflexive? Add the minimum number of ordered pairs to S to make it an equivalence relation. What is the modified S?

   Solution:

   (a). Is S irreflexive?

   A relation is irreflexive if no element in the set is related to itself, i.e., for all $a \in A$, $(a,a) \notin S$. Since none of the pairs $(1,1)$, $(2,2)$, or $(3,3)$ are in S, the relation is irreflexive.

   (b). Add the minimum number of ordered pairs to S to make it an equivalence relation.

   For S to be an equivalence relation, it must satisfy the following properties:

   - **Reflexivity**: For all $a \in A$, $(a,a) \in S$. We need to add the pairs $(1,1)$, $(2,2)$, and $(3,3)$.
   - **Symmetry**: For all $(a,b) \in S$, $(b,a) \in S$. The relation S is already symmetric because $(1,2) \in S$ and $(2,1) \in S$.
   - **Transitivity**: For all $(a,b) \in S$ and $(b,c) \in S$, $(a,c) \in S$. We need to check for transitivity once the reflexive pairs are added.

   Thus, the minimum number of ordered pairs to be added are:

   $$(1,1),(2,2),(3,3)$$

   The modified relation S becomes:

   $$S = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$$

2. Let $m, n \in \mathbb{Z}^+$ with $1 < n \leq m$. Then, $S(m+1,n) = S(m,n-1) + n \cdot S(m,n)$, where $S(m,n)$ denotes a Stirling number of the second kind. Also, prove $O(m,n) = n! \cdot S(m,n)$, where O(m,n) is the number of onto functions in $f : A \to B$.

   Solution:

   Part 1: Proving $S(m+1,n) = S(m,n-1) + n \cdot S(m,n)$

   Consider partitioning $m+1$ elements into n non-empty subsets. We can do this in two ways:

   1) Place the $(m+1)$-th element in a new subset by itself: This leaves m elements to be partitioned into $n-1$ subsets, which can be done in $S(m,n-1)$ ways.

   2) Place the $(m+1)$-th element into one of the existing subsets: First, partition m elements into n subsets ($S(m,n)$ ways), then choose one of these n subsets to place the $(m+1)$-th element (can be done in n ways).

   By the addition principle, the total number of ways is the sum of these two cases:

   $S(m+1,n) = S(m,n-1) + n \cdot S(m,n)$

   Thus, the recurrence relation is proved.

   Part 2: Proving $O(m,n) = n! \cdot S(m,n)$

   We can also prove this by considering the relationship between onto functions and surjections:

1) Every onto function is a surjection.

2) For each partition of A into n subsets (corresponding to the preimages of elements in B), there are $n!$ ways to assign these subsets to elements of B.

3) The number of ways to partition A into n subsets is given by $S(m, n)$.

Therefore, the total number of onto functions is:

$O(m, n) = S(m, n) \cdot n!$

Thus, we have proved that $O(m, n) = n! \cdot S(m, n)$.

3. Give an example of a poset A and a non-empty subset S of A such that S has lower bounds in A, but the greatest lower bound(S) does not exist.

   Solution: Take $A = \mathbb{Q}$ under the standard $\leq$ on rational numbers. Also take $S = \{x \in \mathbb{Q} : x^2 > 2\}$. Every rational number $< \sqrt{2}$ is a lower bound on S. Since $\sqrt{2}$ is irrational, $glb(S)$ does not exist.

   Another example: Take A to be the set of all irrational numbers between 1 and 5, and S to be the set of all irrational numbers between 2 and 3.

   A simpler (but synthetic) example: Take $A = \{a, b, c, d\}$ and the relation on A as,

   $\rho = \{(a, a), (a, c), (a, d), (b, b), (b, c), (b, d)(c, c), (d, d)\}$

   The subset $S = \{c, d\}$ of A has two lower bounds a and b, but these bounds are not comparable to one another.

4. How many permutations of the 10 digits either begin with the 3 digits 987, contain the digits 45 in the fifth and sixth positions, or end with the 3 digits 123?

   Solution: We need to use inclusion-exclusion with three sets. There are 7! permutations that begin 987, since there are 7 digits free to be permuted among the last 7 spaces (we are assuming that it is meant that the permutations are to start with 987 in that order, not with 897, for instance). Similarly, there are 8! permutations that have 45 in the fifth and sixth positions, and there are 7! that end with 123. (We assume that the intent is that these digits are to appear in the order given.) There are 5! permutations that begin with 987 and have 45 in the fifth and sixth positions; 4! that begin with 987 and end with 123; and 5! that have 45 in the fifth and sixth positions and end with 123. Finally, there are 2! permutations that begin with 987, have 45 in the fifth and sixth positions, and end with 123 (since only the 0 and the 6 are left to place). Therefore the total number of permutations meeting any of these conditions is 7! + 8! + 7! - 5! - 4! - 5! + 2! = 50,138.

5. Define a relation ρ on $A = \mathbb{Z} \times \mathbb{N}$ as $(a, b)\rho(c, d)$ if and only if $ad = bc$. Prove that ρ is an equivalence relation. Argue that A/ρ is essentially the set $\mathbb{Q}$ of rational numbers. In abstract algebra, we say that $\mathbb{Q}$ is the field of fractions of the integral domain $\mathbb{Z}$. The equivalence class $[(a, b)]$ is conventionally denoted by a/b.

   Solution:

   1. **Proving ρ is an Equivalence Relation:**

   a) <u>Reflexivity:</u> We need to show $(a, b)\rho(a, b)$ for all $(a, b) \in A$.

   Proof: For any (a, b), we have $ab = ba$ (commutativity of multiplication). Therefore, $(a, b)\rho(a, b)$ holds for all $(a, b) \in A$.

   b) <u>Symmetry:</u> If $(a, b)\rho(c, d)$, we need to show $(c, d)\rho(a, b)$.

   Proof: Given $(a, b)\rho(c, d)$, we know $ad = bc$. Multiplying both sides by 1 (which equals $\frac{1}{cd}$ since $c, d \neq 0$): $ad \cdot \frac{1}{cd} = bc \cdot \frac{1}{cd}$ $\frac{a}{c} = \frac{b}{d}$ Cross-multiplying: $da = cb$ Therefore, $(c, d)\rho(a, b)$.

   c) <u>Transitivity:</u> If $(a, b)\rho(c, d)$ and $(c, d)\rho(e, f)$, we need to show $(a, b)\rho(e, f)$.

   Proof: Given $(a, b)\rho(c, d)$ and $(c, d)\rho(e, f)$, we have: $ad = bc$ and $cf = de$ Multiplying these equations: $ad \cdot cf = bc \cdot de$ $acdf = bcde$ Dividing both sides by cd (note $c, d \neq 0$ as $d \in \mathbb{N}$): $af = be$ Therefore, $(a, b)\rho(e, f)$.

   Thus, ρ satisfies all three properties and is an equivalence relation on A.

2. **Connection to Rational Numbers:**

The equivalence classes of A/ρ are of the form: $[(a,b)] = \frac{a}{b} = \{\frac{an}{bn} \mid n \in \mathbb{N}\}$

This representation shows that each equivalence class corresponds to a rational number $\frac{a}{b}$, and includes all fractions equivalent to $\frac{a}{b}$.

For example, the equivalence class $[(2,3)]$ includes $(2,3)$, $(4,6)$, $(6,9)$, etc., all representing the rational number $\frac{2}{3}$.

A unique representative from each equivalence class can be chosen as $\frac{a}{b}$ with $\gcd(a,b) = 1$. This is the reduced form of the fraction, ensuring each rational number is represented uniquely.

For instance, in the equivalence class of $\frac{2}{3}$, we choose $(2,3)$ as the representative because $\gcd(2,3) = 1$.