Discrete Structures 2024

Pigeonhole Principle - Practice Problems and Solutions

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1. You pick six points in a 3×4 rectangle. Prove that two of these points must be at a distance $\leq \sqrt{5}$.

Solution: Divide the rectangle into 5 parts and then apply PHP. One way of partitioning: connect (1,0), (2,1),(1,2), and then (1,2),(2,3),(1,4) (Considering the lowest point in the leftmost corner is (0,0)). Five partitions: 2 quadrilaterals and 3 Pentagons

Now we can apply PHP. (5 partitions (boxes) and 6 points (pigeons).

2. Let p be a prime number, and x an integer not divisible by p. Prove that there exist non-zero integers a and b of absolute values less than $\sqrt{p}$ such that p divides $ax - b$.

Let p be a prime and x an integer not divisible by p. We are tasked to show that there exist non-zero integers a and b of absolute values less than $\sqrt{p}$ such that $p \mid (ax - b)$.

Consider the integers a and b with $a, b \in \{-\lfloor\sqrt{p}\rfloor, \ldots, \lfloor\sqrt{p}\rfloor\}$. There are $2\lfloor\sqrt{p}\rfloor + 1$ possible values for both a and b, and hence, there are $(2\lfloor\sqrt{p}\rfloor + 1)^2$ pairs (a, b).

Now, consider the values of $ax - b \mod p$ for each pair. There are at most p distinct values for $ax - b \mod p$.

Since the number of pairs (a, b) exceeds p (i.e., $(2\lfloor\sqrt{p}\rfloor + 1)^2 > p$, by the pigeonhole principle, there must exist distinct pairs (a_1, b_1) and (a_2, b_2) such that:

$$a_1 x - b_1 \equiv a_2 x - b_2 \mod p.$$

This implies:

$$(a_1 - a_2)x \equiv b_1 - b_2 \mod p.$$

Since x is not divisible by p, it follows that $a_1 \neq a_2$, so p divides $a_1 x - a_2 x - (b_1 - b_2)$. Therefore, we have found integers $a = a_1 - a_2$ and $b = b_1 - b_2$, both non-zero, with absolute values less than $\sqrt{p}$, such that $p \mid (ax - b)$.

3. Let $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$. Use the pigeonhole principle to prove that $ua + vb = 1$ for some $u, v \in \mathbb{Z}$.

Solution:

$$\gcd(a, b) = 1$$

Consider the set $S = \{a, 2a, \ldots, (b-1)a\}$.

Take mod b, we get $S' = \{r_1, \ldots, r_{b-1}\}$.

Observe that remainder 0 does not occur.

Now, assume that remainder 1 does not occur either. So we can apply the Pigeonhole Principle (since there are $b-1$ remainders and only $b-2$ possible values):

Thus, we can find positive integers m, n, where $0 < m < n < b$ such that:

$$ma \equiv na \pmod{b}$$

But since $\gcd(a, b) = 1$, it follows that $b \mid (n - m)$. This is a contradiction because $0 < n - m < b$.

Thus, there exists a u such that:

$$ua \equiv 1 \pmod{b}$$

which implies

$$ua + vb = 1$$

for some integer v.

4. Let p be a prime number, and x an integer not divisible by p. Now assume that p is of the form $4k + 1$. We know from number theory that in this case, there exists an integer x such that p divides $x^2 + 1$. Show that $p = a^2 + b^2$ for some integers a, b.

Sol^n:- $\quad p = 4K+1$, $\quad p \mid x^2+1$

$\Rightarrow x^2 \equiv -1 \pmod p$

$p = a^2+b^2 \Rightarrow a, b \in \{0, 1, \dots m\}$ with $m = \lfloor \sqrt{p} \rfloor$

Consider $a + x b$ for all the pairs $[(m+1)^2 > p]$

$\downarrow$ mod p

$r_1, r_2, \dots r_{(m+1)^2}$

From $\boxed{PHP}$ two such remainders will be the same

let $a_1 + x b_1 \equiv a_2 + x b_2 \pmod p$

$$\Rightarrow \underbrace{(a_1 - a_2)}_{y} \equiv -x \underbrace{(b_1 - b_2)}_{z} \pmod p$$

$$\Rightarrow y \equiv -xz \pmod p$$

$$\Rightarrow y^2 \equiv +x^2 z^2 \pmod p$$

$x^2 \to \equiv -1 \pmod p$

$$\Rightarrow y^2 \equiv -z^2 \pmod p$$

$$\Rightarrow p \mid y^2 + z^2$$

but $y = a_1 - a_2 \Rightarrow |y| \leq m$, $|z| \leq m$

$$\Rightarrow 0 < y^2+z^2 \leq 2m^2 < 2p$$

$$\Rightarrow y^2+z^2 = p$$

5. Let $n \geq 10$ be an integer. You choose n distinct elements from the set $\{1, 2, 3, \dots, n^2\}$. Prove that there must exist two disjoint subsets of the chosen numbers whose sums are

equal.

The sum of the elements of a subset of $\{1, 2, 3, \ldots, n^2\}$ of size less than n is $< n^3$. The chosen collection has $2^n - 1$ non-empty subsets. For $n \geq 10$, we have $2^n - 1 > n^3$, so by PHP, there must exist two different non-empty subsets A and B of the chosen numbers such that $\sum_{a \in A} a = \sum_{b \in B} b$. If A and B are not disjoint, take $A - (A \cap B)$ and $B - (A \cap B)$ as A and B.

6. Let $p(x)$ be a polynomial with integer coefficients, having three distinct integer roots a, b, c. Prove that the polynomials $p(x) \pm 1$ cannot have any integer roots.

   Suppose that an integer d exists with $p(d) \pm 1 = 0$, that is, with $p(d) = \pm 1$. Clearly, d is different from a, b, c. For all $u, v \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, we have $(u - v) | (u^n - v^n)$, and so $(u - v) | (p(u) - p(v))$. But then, the non-zero differences $d - a, d - b, d - c$ all divide $\pm 1 - 0 = \pm 1$, and so can only be ± 1. Therefore, at least two of the three differences $d - a, d - b, d - c$ must be the same, contradicting the fact that a, b, c are distinct from one another.

7. 65 distinct integers are chosen in the range $1, 2, 3, \ldots, 2022$. Prove that there must exist four of the chosen integers (call them a, b, c, d) such that $a - b + c - d$ is a multiple of 2022.

   The total count of 2-subsets of the 65 chosen integers is $\binom{65}{2} = 2080 > 2022$. So we can find two distinct subsets $S = \{a, c\}$ and $T = \{b, d\}$ of the chosen integers such that $(a + c) \operatorname{rem} 2022 = (b + d) \operatorname{rem} 2022$, that is, $(a - b + c - d) \operatorname{rem} 2022 = 0$ (where rem means remainder of Euclidean division). We need to show that $S \cap T = \emptyset$. Suppose not. Since S and T are distinct, we must have $|S \cap T| = 1$. Say, $a = b$ (but $c \neq d$). But then, the condition $(a + c) \operatorname{rem} 2022 = (b + d) \operatorname{rem} 2022$ implies that $c \operatorname{rem} 2022 = d \operatorname{rem} 2022$. But c and d are chosen in the range $[1, 2022]$, so they must be equal, a contradiction.

Discrete Structures
Tutorial Solutions
Pigeonhole Principle

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1. Prove that in any group of 10 distinct positive integers between 1 and 50, there are at least two numbers whose difference is at most 5.

Solution:
Consider dividing the numbers 1 to 50 into 9 groups as follows:

$$\{1, 2, 3, 4, 5, 6\}, \{7, 8, 9, 10, 11, 12\}, \ldots, \{43, 44, 45, 46, 47, 48\}, \{49, 50\}.$$

Each group has either 6 or 2 elements, so there are 9 groups in total. Now, if you select 10 distinct numbers, by the pigeonhole principle, at least two of those numbers must come from the same group.
Since the maximum difference between any two numbers within a single group is 5 (for example, the difference between 1 and 6 in the first group), there must be at least two numbers among the selected 10 whose difference is at most 5.

[Tutorial Q2] A repunit is an integer of the form 111 ... 1. Prove that any $n \in N$ with $gcd(n, 10) = 1$ divides a repunit.

Solution: Note that, a repunit is a number of the form 111 ... 1, which can also be represented as:

$$R_k = \frac{10^k - 1}{9},\ k \in N$$

Let's consider the sequence $10^1, 10^2, 10^3\ mod\ 9n$. This sequence can take on a limited number of possible values since there are only 9n different remainders possible when dividing by 9n (i.e., 0, 1, 2,...,9n-1).

By the Pigeonhole Principle, after considering 9n + 1 terms in the sequence $10^1 mod\ n, 10^2\ mod\ n, 10^3\ mod\ n,\ ....,\ 10^{9n+1} mod\ 9n$, at least two of these must be the same (because there are 9n + 1 terms but only 9n possible remainders).

Now, Suppose $10^i \equiv 10^j\ (mod\ 9n)$ for some $i < j$. This means:

$$10^j - 10^i \equiv 0\ (mod\ 9n)$$

Or

$$10^i(10^{j-i} - 1) \equiv 0\ (mod\ 9n)$$

Since $gcd(n,\ 10) = 1 = gcd(9n, 10)$, $9n \mid (10^{j-i} - 1)$ .Therefore, $n \mid R_{j-i} = \frac{10^{j-i}-1}{9}$.

[Tutorial Q3] Show that there exists an integer n such that $0 < sin\, n < 2^{-2022}$.

Solution: Consider the function $cos\,x$. Divide the interval $[-1, 1]$ into $(2^{2022})^2$ equal partitions. This gives us the partition size of $\frac{2}{(2^{2022})^2} = 2(2^{-2022})^2$.

Now, consider $2(2^{2022})^2 + 1$ elements $cos\,1,\ cos\,2,\ cos\,3, \ldots, cos(2(2^{2022})^2 + 1)$. Atleast 1 partition will have 3 elements by Pigeonhole Principle. Let them be $cos\,x$, $cos\,y$ and $cos\,z$ Now, atleast 2 of x, y, z will have the same parity. Without loss of generality, let those elements be x and y, with $cos\,x > cos\,y$. So, we have

$$0 < cos\,x - cos\,y < 2(2^{-2022})^2$$
$$0 < 2\,sin\,(\tfrac{x+y}{2})\,sin(\tfrac{y-x}{2}) < 2(2^{-2022})^2$$
$$0 < sin(\tfrac{x+y}{2})\,sin(\tfrac{y-x}{2}) < (2^{-2022})^2$$

Now $(\frac{x+y}{2})$ and $(\frac{x-y}{2})$ are integers since x and y have the same parity. Now,

$$0 < ab < k^2$$
$$\Rightarrow 0 < min(|a|, |b|) < k \qquad \text{(trivial)}$$

So, one of $n = \frac{x+y}{2}$ or $n = \frac{x-y}{2}$ satisfies the inequality denoted by $0 < |sin\,n| < 2^{-2022}$. If $0 < sin\,n < 2^{-2022}$, n is the required integer. Otherwise, if $0 > sin\,n > -2^{-2022}$ then $0 < sin\,(-n) < 2^{-2022}$ and $-n$ is the required integer.

[Tutorial Q4] Let ξ be an irrational number. Prove that given any real ε > 0 (no matter how small), there exist infinitely many pairs of integers a, b such that $0 < a\xi - b < \varepsilon$.

Solution: Consider the irrational number ξ. Since, ξ is irrational, it cannot be expressed as a ratio of two integers. For any ε > 0, fix a positive number N such that $N > 1/\varepsilon$. Define N+1 numbers of the form $\{k\xi\}$ for $k = 0, 1, 2, \ldots, N$, where $\{x\}$ denotes the fractional part of x. The fractional parts $\{k\xi\}$ are real numbers in the interval $[0, 1)$. By PHP, since there are N+1 fractional parts but only N subintervals of length $1/N$ that partition in the interval $[0, 1)$, at least two of these fractional parts, say $\{m\xi\}$ and $\{n\xi\}$ with m > n, must lie in the same subinterval. Thus, we have $|\{m\xi\} - \{n\xi\}| < 1/N$. Since $\{m\xi\} - \{n\xi\} = (m-n)\xi - \lfloor (m-n)\xi \rfloor$, there exist integers $a = m - n$ and $b = \lfloor (m-n)\xi \rfloor$ such that:

$$\lfloor a\xi - b \rfloor < 1/N$$

Notice that $a = m - n > 0$ and $1/N < \varepsilon$ by the choice of N. Therefore, $0 < a\xi - b < \varepsilon$. Since there are infinitely many choices of N, we have infinitely many pairs (a, b) that satisfy the given condition.

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5. Let $n \geq 2$ be an integer. You choose n distinct integers from the set $\{1, 2, 3, \ldots, n^2 - 1\}$. Prove that there must be two of the chosen integers (call them x and y) satisfying $0 < \sqrt{x} - \sqrt{y} < 1$.

This follows from a direct application of the pigeon-hole principle.
The $n-1$ holes are $\{1, 2, 3\}, \{4, 5, 6, 7, 8\}, \{9, 10, 11, 12, 13, 14, 15\}, \ldots, \{(n-1)^2, (n-1)^2 + 1, (n-1)^2 + 2, \ldots, n^2 - 1\}$.
The pigeons are the n chosen integers.