CS21201: Discrete Structures

Tutorial II : Logic

Question 1

Let

GG: Guard of gold road tells the truth
GM: Guard of marble road tells the truth
GS: Guard of stone road tells the truth
G: Gold road leads to the centre
M: Marble road leads to the centre
S: Stone road leads to the centre

Statements

$$GG \longleftrightarrow G \wedge (S \to M)$$
$$GM \longleftrightarrow \neg G \wedge \neg S$$
$$GS \longleftrightarrow G \wedge \neg M$$

Since no guard tells the truth, the following statement is true.

$$\begin{aligned} Z &\equiv \neg GG \wedge \neg GM \wedge \neg GS \wedge \neg[G \wedge \neg M] \\ &\equiv \neg[G \wedge (S \to M)] \wedge \neg[\neg G \wedge \neg S] \wedge \neg[G \wedge \neg M] \\ &\equiv [\neg G \vee (S \wedge \neg M)] \wedge [G \vee S] \wedge [\neg G \vee M] \end{aligned}$$

G	M	S	$\neg G \vee (S \wedge \neg M)$	$G \vee S$	$\neg G \vee M$	Z
0	0	0	1	0	1	0
0	0	1	1	1	1	1
0	1	0	1	0	1	0
0	1	1	1	1	1	1
1	0	0	0	1	0	0
1	0	1	1	1	0	0
1	1	0	0	1	1	0
1	1	1	0	1	1	0

$\therefore \boxed{Z \equiv \neg G \wedge S}$

Question 2

rob: murder was done for robbery
pol: murder was political assasciation
wom: murder was for a woman
tak: something was taken from the murderer's place
imm: assassin left immediately
trc: assassin left tracks all over the room

1. $\neg rob \rightarrow pol \vee wom$
2. $rob \rightarrow tak$
3. $\neg tak$
4. $pol \rightarrow imm$
5. ~~pol~~ trc
6. $trc \rightarrow \neg imm$

$rob \rightarrow tak$
$\neg tak$
$\therefore \neg rob$
(Modus Tollens)

$\neg rob$
$\neg rob \rightarrow pol \vee wom$
$\therefore pol \vee wom$
(Modus Ponens)

trc
$trc \rightarrow \neg imm$
$\therefore \neg imm$
(Modus Ponens)

$\neg imm$
$pol \rightarrow imm$
$\therefore \neg pol$
(Modus Tolens)

$pol \vee wom$
$\neg pol$
$\therefore wom$
(Disjunctive Syllogisim)

Thus, it follows that the ~~mur~~ murder was done for a Woman

Question 3

(a)

$$\frac{\neg(s \vee u)}{\therefore \neg s,\ \neg u}$$

$$\frac{\begin{array}{c}\neg u \\ t \to u\end{array}}{\therefore \neg t}$$

$$\frac{\begin{array}{c}r \to (s \vee t) \\ \neg s,\ \neg t\end{array}}{\therefore \neg r}$$

$$\frac{\begin{array}{c}(\neg p \vee q) \to r \\ \neg r\end{array}}{\therefore p,\ \neg q}$$

$$\frac{\begin{array}{c}q \leftrightarrow v \\ \neg q\end{array}}{\therefore \neg v}$$

$$\frac{\begin{array}{c}(v \wedge \neg w) \vee (\neg v \wedge w) \to \neg p \\ p\end{array}}{\therefore (\neg v \vee w) \wedge (v \vee \neg w)}$$

$$\frac{\begin{array}{c}(\neg v \vee w) \wedge (v \vee \neg w) \\ \neg v\end{array}}{\therefore \neg w}$$

(b)

$$\frac{\begin{array}{c}p \to u \\ u \to s\end{array}}{\therefore p \to s}$$

$$\cancel{\frac{\begin{array}{c}p \to s \\ \neg r \to \neg s\end{array}}{\therefore s \to r}}$$

$$\frac{\begin{array}{c}p \to s \\ \neg r \to \neg s \equiv s \to r\end{array}}{\therefore p \to r}$$

$$\frac{\begin{array}{c}p \to r \\ \neg t \to \neg r \equiv r \to t\end{array}}{\therefore p \to t}$$

$$\frac{\begin{array}{c}p \to t \\ t \to q\end{array}}{\therefore p \to q}$$

Question 4

(a) "At most one person is a ruler"

Either there is no ruler, or only a single person is a ruler and no one else.

$$\neg \exists p \left[Person(p) \wedge Ruler(p) \right] \quad \vee$$
$$\exists p \left[Person(p) \wedge Ruler(p) \wedge \forall q \left(Person(q) \wedge (p \neq q) \right) \rightarrow \neg Ruler(q) \right]$$

Alternative

$$\exists p \left[Person(p) \wedge \forall q . \left[Person(q) \wedge Ruler(q) \rightarrow (p = q) \right] \right]$$
$$\vee \quad \forall p . \perp$$ ← FALSE truth value

(b) If time has a beginning, there is an instant that precedes every other instant

If time has no end, for every instant, there will be another that comes after it.

$$\exists x . \left[Instant(x) \wedge \forall y . \left[Instant(y) \wedge (x \neq y) \rightarrow Precedes(x, y) \right] \right]$$

$$\wedge$$

$$\forall x . \left[Instant(x) \rightarrow \exists y . \left[Instant(y) \wedge Precedes(x, y) \right] \right]$$

Question 5

$P(x)$ is an open statement

(a) $P(x)$ is true for exactly one value of x

$$\exists x\,[P(x) \wedge \forall y\,[P(y) \rightarrow (y = x)]]$$

(b) $P(x)$ is true for exactly two values of x

$$\exists x \exists y\,[P(x) \wedge P(y) \wedge (x \neq y) \wedge \forall z\,[P(z) \rightarrow (z = x) \vee (z = y)]]$$

(c) $P(x)$ is true for atmost one value ~~atleast two values~~ of x

$$\forall x\,[\neg P(x)] \vee \exists x\,[P(x) \wedge \forall y\,[P(y) \rightarrow (y = x)]]$$

(d) $P(x)$ is true for atleast two values of x

$$\exists x \exists y\,[P(x) \wedge P(y) \wedge (x \neq y)]$$

(e) If $P(x)$ is true for two values of x, it is true for all values of x

$$\exists x \exists y\,[P(x) \wedge P(y) \wedge (x \neq y)] \rightarrow \forall x\, P(x)$$

(f) If $P(x)$ is true for at least two values of x, it is true for at least three values of x.

$$\exists x \exists y\,[P(x) \wedge P(y) \wedge (x \neq y)] \rightarrow$$
$$\exists x \exists y \exists z\,[P(x) \wedge P(y) \wedge P(z) \wedge (x \neq y) \wedge (y \neq z) \wedge (z \neq x)]$$

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Practice 2: Logic (Solutions)

Question 1

Solution I: Assume that Ronaldo knows Mbappe means that they both know each other. Propositions used are as follows:

K : Mbappe knows Ronaldo (Ronaldo knows Mbappe)
HL : Haaland likes the cookies
MBP : Mbappe was on the pitch
MEP : Messi was on the pitch

Messi: $K \wedge HL$
Mbappe: $\neg K \wedge \neg MBP$
Haaland: $MBP \wedge MEP$

Since one and only one of Haaland, Mbappe or Messi ate the cookies. We break the problem down into three cases depending on who ate the cookies:

a. Messi: Both Mbappe and Haaland are telling the truth. But for that to be true, $\neg MBP$ and MBP have to be true at the same time, **not possible**
b. Haaland: Both Messi and Mbappe are telling the truth. But for that to be true, K and $\neg K$ have to be true, **not possible**
c. Mbappe: It can be seen that this is the only possible case. Since we can keep HL = 1 and MEP = 1.

Solution II: Assume that Ronaldo knows Mbappe but Mbappe does not know Ronaldo. In that case, we design two predicates:
KMR : Mbappe knows Ronaldo
KRM : Ronaldo knows Mbappe
Messi: $KRM \wedge HL$
Mbappe: $\neg KMR \wedge \neg MBP$
Haaland: $MBP \wedge MEP$

Since one and only one of Haaland, Mbappe or Messi ate the cookies. We break the problem down into three cases depending on who ate the cookies:

a. Messi: Not possible, see **Solution I**
b. Haaland: Both Messi and Mbappe are telling the truth. But for that to be true, KRM and $\neg KMR$ have to be true (in addition to HL and $\neg MBP$), **possible**

c. <u>Mbappe:</u> Possible, see **Solution I**

Under this assumption, the identity of the thief cannot be ascertained.

<u>Question 2</u>

Denote the $NAND(x)$ function as $N(x)$. You can write $\neg X$ as $N(X, X)$.

a. $P \rightarrow Q \equiv \neg P \vee Q \equiv \neg(P \wedge \neg Q)$
$\equiv N(P, \neg Q)$ (Using De Morgan's Theorem)
$\equiv N(P, N(Q, Q))$

b. $P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$
$\equiv N(P, N(Q, Q)) \wedge N(Q, N(P, P))$

To simplify things a bit, choose:
$X = N(P, N(Q, Q))$ and $Y = N(Q, N(P, P))$

$P \leftrightarrow Q \equiv X \wedge Y \equiv \neg(\neg X \vee \neg Y)$ (using De Morgan's Theorem)
$\equiv N(N(X, X), N(Y, Y))$

c. $P \oplus Q \equiv (P \vee Q) \wedge (\neg Q \vee \neg P)$
$X = (P \vee Q) \equiv N(\neg P, \neg Q) \equiv N(N(P, P), N(Q, Q))$
$Y = (\neg P \vee \neg Q) \equiv N(P, Q)$
Repeat the solution to part (b).

Question 3

Q3

a. $p \vee (q \wedge r)$

$p \rightarrow s \equiv \neg p \vee s$

$\therefore (q \wedge r) \vee s \equiv (q \vee s) \wedge (r \vee s)$

$\therefore r \vee s$ [If multiple conditions are joined by $\cdot \wedge$, each are individually concluded true]

(proved)

b. $\neg p \vee s$

$p \vee q \vee t$

$\therefore s \vee q \vee t$

$s \vee q \vee t$

$\neg t \vee (s \wedge r)$

$\therefore s \vee q \vee (s \wedge r)$

$\equiv (s \vee q) \wedge (s \vee q \vee r)$

$s \vee q$

$\neg q \vee r$

$\therefore r \vee s$ (proved)

c. $\neg j \vee k$

$\neg k$

$\therefore \neg j$

$k \vee l$

$\neg k$

$\therefore l$

Since $\neg j$ and l are both true,

$\neg j$, $\neg l$

$(l \wedge \neg j) \rightarrow (m \wedge \neg j)$

$\therefore m$, $\neg j$

(proved)

d. $r \to m \equiv \neg r \vee m$ $\quad$ $\neg m, \neg n$

$$\frac{\neg m}{\therefore \neg r} \qquad \frac{(\neg m \wedge \neg n) \to (o \to n)}{\therefore o \to n \equiv \neg o \vee n}$$

$$\frac{\begin{array}{l}\neg o \vee n \\ \neg n\end{array}}{\therefore \neg o \text{ (proved)}}$$

e. $w \to x \equiv \neg w \vee x$

$$\frac{(w \wedge x) \to y \equiv \neg w \vee \neg x \vee y}{\therefore \neg w \vee y}$$

$$\frac{\begin{array}{l}\neg w \vee y \\ (w \wedge y) \to z \equiv \neg w \vee \neg y \vee z\end{array}}{\therefore \neg w \vee z \equiv w \to z \text{ (proved)}}$$

f. $$\frac{\begin{array}{l}p \vee s \\ \neg s\end{array}}{\therefore p} \qquad \frac{\begin{array}{l}t \to q \equiv \neg t \vee q \\ t \vee u\end{array}}{\therefore q \vee u}$$

$$\frac{\begin{array}{l}p \\ p \to (q \to r) \equiv \neg p \vee (q \to r)\end{array}}{\therefore q \to r \equiv \neg q \vee r}$$

$$\frac{\begin{array}{l}q \vee u \\ \neg q \vee r\end{array}}{\therefore u \vee r \text{ (proved)}}$$

Question 4

Coding using Propositional Logic is as follows:

S1 : $\neg S \vee D$

S2 : $\neg R \vee E$

S3 : $(S \wedge \neg R) \vee (\neg S \wedge R)$

S4 : $\neg S \vee E$

S5 : $\neg R \vee D$

G : $(\neg E \vee \neg D) \wedge (D \vee E)$

The goal should be easy enough to derive using the truth table method

Question 5

Q5

(a)

p	q	r	$p\wedge\neg q$	$p\to q\to r$ $\neg p\vee\neg q\vee r$	$\neg r$
0	0	0	0	1	1
0	0	1	0	1	0
0	1	0	0	1	1
0	1	1	0	1	0
1	0	0	1	1	1
1	0	1	1	1	0
1	1	0	0	0	1
1	1	1	0	1	0

$(p\wedge\neg q)$ and $[p\to(q\to r)]$ both are true, but $\neg r$ is false.

(b)

p	q	r	$(p\wedge q)\to r$ $\neg p\vee\neg q\vee r$	$\neg q\vee r$	p
0	0	0	1	1	0
0	0	1	1	1	0
0	1	0	1	0	0
0	1	1	1	1	0
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	1
1	1	1	1	1	1

Multiple counter-examples.

(c)

p	q	r	s	$p \leftrightarrow q$	$q \to r$	$s \to r$ ($\equiv r \vee \neg s$)	$\neg s \to \neg q$ ($\equiv s \vee \neg q$)	s
0	0	0	0	1	1	1	1	0
0	0	0	1	1	1	0	1	1
0	0	1	0	1	1	1	1	0
0	0	1	1	1	1	1	1	1
0	1	0	0	0	0	1	0	0
0	1	0	1	0	0	0	1	1
0	1	1	0	0	1	1	0	0
0	1	1	1	0	1	1	1	1
1	0	0	0	0	1	1	1	0
1	0	0	1	0	1	0	1	1
1	0	1	0	0	1	1	1	0
1	0	1	1	0	1	1	1	1
1	1	0	0	1	0	1	0	0
1	1	0	1	1	0	0	1	1
1	1	1	0	1	1	1	0	0
1	1	1	1	1	1	1	1	1

Counter-examples

<u>Question 6</u>

(a) **J:** Jonas goes to the meeting
C: Complete report is made
E: Special election held
I: Investigation is launched
T: Members stand trial
D: Organization disintegrates

If Jonas goes to the meeting, then a complete report will be made

$J \rightarrow C$

If Jonas does not go to the meeting, then a special election will be required

$\neg J \rightarrow E$

If a complete report is made, then an investigation will be launched

$C \rightarrow I$

If Jonas's going to the meeting implies that a complete report will be made, and the making of a complete report implies that an investigation will be launched, then either Jonas goes to the meeting and an investigation is launched or Jonas does not go to the meeting and no investigation is launched

$(J \rightarrow C) \wedge (C \rightarrow I) \rightarrow (J \wedge I) \vee (\neg J \wedge \neg I)$

If Jonas goes to the meeting and an investigation is launched, then some members will have to stand trial

$(J \wedge I) \rightarrow T$

If Jonas does not go to the meeting and no investigation is launched, then the organization will disintegrate very rapidly

$(\neg J \wedge \neg I) \rightarrow D$

Therefore either some members will have to stand trial or the organization will disintegrate very rapidly

$T \wedge D$

I am not explicitly solving this since the conclusion is obvious. Solving each of these statements one by one by following the implication logic will lead to the last conclusion.

(b) SN: Mr. Smith is the manager's next-door neighbor
SH: Mr. Smith lives halfway between Detroit and Chicago
SC: Mr. Smith lives in Chicago
RD: Mr. Robinson lives in Detroit

RC: Mr. Robinson lives in Chicago
JC: Mr. Jones lives in Chicago
JM: Mr. Jones is the manager

If Mr. Smith is the manager's next-door neighbor, then Mr. Smith lives halfway between Detroit and Chicago.

$SN \rightarrow SH$

If Mr. Smith lives halfway between Detroit and Chicago, then he does not live in Chicago.

$SH \rightarrow \neg SC$

Mr. Smith is the manager's next-door neighbor.

SN

If Mr. Robinson lives in Detroit, then he does not live in Chicago.

$RD \rightarrow \neg RC$

Mr. Robinson lives in Detroit.

RD

Mr. Smith lives in Chicago or else either Mr. Robinson or Mr. Jones lives in Chicago.

$SC \oplus (RC \vee JC)$

If Mr. Jones lives in Chicago, then the manager is Jones.

$JC \rightarrow JM$

Therefore the manager is Jones.

JM

$SN \rightarrow SH$
SN
___
$\therefore SH$

$SH \rightarrow \neg SC$
SH
___
$\therefore \neg SC$

$RD \rightarrow \neg RC$
RD
___
$\therefore \neg RC$

$SC \oplus (RC \vee JC)$
$\neg SC$
___
$\therefore RC \vee JC$

$RC \vee JC$
$\neg RC$
___
$\therefore JC$

$JC \rightarrow JM$
JC
___
$\therefore JM$ (proved)

Question 7

1. Coding of the statements is as under ($x \in$ Creatures)

   S1: $\forall x\ [Lion(x) \rightarrow Fierce(x)]$

   S2: $\exists x\ [Lion(x)\ \wedge\ \neg DrinksC(x)]$

   a. $\exists x\ Fierce(x)$
   This statement is true if there is at least one lion. We cannot say that this directly follows from S1. This statement is **False.**

   b. From S2, we notice that $\exists x\ [Lion(x)\ \wedge\ \neg DrinksC(x)]$. So let that creature be p. We know that p is a lion. From S1, any creature who is a lion is fierce. Therefore p is fierce. By existential generalization, $\exists x\ Fierce(x)$. Hence this statement is **True.**

c. $\exists x\ [Fierce(x)\ \wedge\ \neg DrinksC(x)]$: Notice that similar to (b), we derive that p is a lion and p does not drink coffee. From S1, all lions are fierce. Therefore p is fierce. This implies that $Fierce(p)\ \wedge\ \neg DrinksC(p)$, by existential generalization, $\exists x\ [Fierce(x)\ \wedge\ \neg DrinksC(x)]$. The statement is **True.**

Question 8

(a) Predicates used are $Respect(x,\ y)$: person x respects person y and $Hire(x,\ y)$: person x hires person y.

S1 : $\forall x\ (\neg Respect(x,\ x)\ \rightarrow \neg\exists y(Respect(y,\ x))$

S2 : $\forall x \forall y\ (\neg Respect(x,\ y)\ \rightarrow \neg Hire(x,\ y))\ \equiv\ \forall x \forall y\ (Hire(x,\ y)\ \rightarrow Respect(x,\ y))$

G : $\forall x\ [(\neg\exists y\ Respect(x,\ y))\ \rightarrow (\neg\exists z\ Hire(z,\ x))]$

Simplification of G:

$\forall x\ [(\neg\exists y\ Respect(x,\ y))\ \rightarrow (\neg\exists z\ Hire(z,\ x))]$ (1)

$\forall x\ [(\exists z\ Hire(z,\ x))\ \rightarrow (\exists y\ Respect(x,\ y))]$ (2) Contrapositive (1)

` Proof by contradiction, assume that $\neg G$ is true.

$\neg\forall x\ [(\exists z\ Hire(z,\ x))\ \rightarrow (\exists y\ Respect(x,\ y))]$ (3)

$\exists x\ \neg[\neg(\exists z\ Hire(z,\ x))\ \vee\ (\exists y\ Respect(x,\ y))]$ (4) Properties of $\neg$ and $\rightarrow$

$\exists x\ [(\exists z\ Hire(z,\ x))\ \wedge\ \neg(\exists y\ Respect(x,\ y))]$ (5) De Morgan's Laws

Instantiate (5), by $x\ =\ A$ and $z\ =\ B$

$Hire(B,\ A)$ (6)

$\neg(\exists y\ Respect(A,\ y))$ (7)

Instantiate S2 by $x\ =\ B$ and $z\ =\ A$ (8)

$Hire(B,\ A)\ \rightarrow\ Respect(B,\ A)$

$Respect(B,\ A)$ (9) Modus Ponens (7, 8)

$\forall x(\ \exists y\ (Respect(y,\ x))\ \rightarrow Respect(x,\ x))$ (10) Contrapositive (S1)

Instantiate by $x\ =\ A$ and $y\ =\ B$

$Respect(B,\ A)\ \rightarrow\ Respect(A,\ A)$ (11)

$Respect(A,\ A)$ (12) Modus Ponens(9, 11)

But from (7), $\neg(\exists y\ Respect(A,\ y))\ \Rightarrow\ \forall y\ \neg Respect(A,\ y) \Rightarrow \neg Respect(A,\ A)$

Hence we have a contradiction

(b) Predicates:

$a(x)$: Person x belongs to the Alpine Club

$s(x)$: Person x is a skier

$m(x)$: Person x is a mountain climber

$l(x,\ y)$: Person x likes weather event y

Statements:

S1 : $a(Tony)\ \wedge\ a(Mike)\ \wedge\ A(John)$
S2 : $\forall x\ [a(x)\ \rightarrow\ (s(x)\ \vee\ m(x))]$
S3 : $\neg\exists x\ [m(x)\ \wedge\ l(x,\ Rain)]$
S4 : $\forall x[s(x)\ \rightarrow l(x,\ Snow)]$
S5 : $\forall y[l(Mike,\ y) \leftrightarrow \neg l(Tony,\ y)]$
S6 : $l(Tony,\ Rain)\ \wedge\ l(Tony,\ Snow)$

Since Tony likes both Rain and Snow and Mike dislikes whatever Tony likes and likes whatever Tony dislikes
Mike does not like Rain and Mike does not like Snow

$\neg l\ (Mike,\ Rain)$ (1)
$\neg l\ (Mike,\ Snow)$ (2)

From S4, instantiating $x\ =\ Mike$, we get

$s(Mike)\ \rightarrow l(Mike,\ Snow)$ (3)
$\neg s(Mike)$ (4) Modus Tollens(2, 3)

From S2, instantiating $x\ =\ Mike$, we get

$a(Mike)\ \rightarrow (s(Mike)\ \vee\ m(Mike))$ (5)
$s(Mike)\ \vee\ m(Mike)$ (6) Modus Ponens(S1, 5)
$m(Mike)$ (7) (4, 6)

Clearly **Mike** is a Mountain Climber and not a skier, from (4) and (7).