CS21201 Discrete Structures
Tutorial & Practice Problems Solution
Elementary Counting Techniques

Problem 1

E → 3
N → 3
G → 2
I → 2
R → 1

1 (a) $\frac{11!}{3! + 3! + 2! + 2!}$

(b) $\frac{9!}{3! + 2! + 2!}$

(c) | — | — | — | — | — | — | — | — |

9 places (|) for E
8 places (—) for other

$= {}^9C_3 * \frac{8!}{3!\ 2!\ 2!}$

Alt: Total − (E's never together) → neither 2 E's nor 3 E's

$= \frac{11!}{3!\ 3!\ 2!\ 2!} - \frac{9! + 9}{3!\ 2!\ 2!}$

(d) E $\underbrace{\qquad\qquad\qquad}_{9}$ G

$\frac{9!}{\underset{E}{2!} \cdot \underset{N}{3!}\ \underset{I}{2!}}$

(e) Put all E's followed by all N's (There can be 0 or more letters in place of |)

| E | E | E | N | N | N |

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 5 \Rightarrow \binom{7+5-1}{5} * \frac{5!}{2!\ 2!}$

(f) All vowels Adjacent

(E E E I I) | $\underbrace{\qquad\qquad}_{6}$
together

$= \frac{7!}{3!\ 2!} * \frac{5!}{3!\ 2!}$

Problem 2

No of times the statements inside the innermost for loop is executed

$$= \binom{15+3-1}{3} = \binom{17}{3} = 680$$

Now, after execution of this segment, the value of

$$\text{counter} = \sum_{i=1}^{680} i = \frac{680 \times 681}{2}$$

Problem 3

Out of n $\quad \left\lceil \frac{n}{2} \right\rceil$ odd $\quad \left\lfloor \frac{n}{2} \right\rfloor$ even

choose k elemats

$\#$ odds $\left\lfloor \frac{k}{2} \right\rfloor + 1, \ldots k$

$$\sum_{x=1}^{k-\lfloor \frac{k}{2} \rfloor} \binom{\lceil \frac{n}{2} \rceil}{\lfloor \frac{k}{2} \rfloor + x} \binom{\lfloor \frac{n}{2} \rfloor}{k - \lfloor \frac{k}{2} \rfloor - x}$$

Problem 4

3 choices (6, 7, 9)

If first digit is "6"

Remaining digit can be placed in any order, but 5 is repeated twice

$\Rightarrow \frac{6!}{2!} = 360$

If first digit is "7 or 9"

In this case 5 as well as 6 is repeated twice

$\Rightarrow 2 * \frac{6!}{2! \cdot 2!} = 360$

Total = 360 + 360 = 720

Problem 5

If at all points, there are atleast as many 100 Rs. notes as 200 Rs. notes, you'll not have to give a slip to anyone.

Answer: C(75)

Problem 6

Committee $\boxed{18}$

Men: 15
Women: 20

(a) $\binom{15+20}{18} = \binom{35}{18}$

(b) $\binom{15}{8}\cdot\binom{20}{6}\cdot\binom{21}{4}$

$15-8 = 7$
$20-6 = 14$
$8+6 = 14$

(c) No of Men

0, 2, 4, 6, 8, 10, 12, 14

$$\binom{15}{0}\cdot\binom{20}{18} + \binom{15}{2}\cdot\binom{20}{16} + \binom{15}{4}\cdot\binom{20}{14} + \cdots + \binom{15}{14}\binom{20}{4}$$

$$= \sum_{i=0}^{7} \binom{15}{2i}\cdot\binom{20}{18-2i}$$

(d) More men than women

No of Men: 10, 11, 12, 13, 14, 15.

$$\binom{15}{10}\cdot\binom{20}{8} + \binom{15}{11}\cdot\binom{20}{7} + \cdots + \binom{15}{15}\cdot\binom{20}{3}$$

$$= \sum_{i=10}^{15} \binom{15}{i}\cdot\binom{20}{18-i}$$

(e) at least 8 men

$$\sum_{i=8}^{15} \binom{15}{i}\cdot\binom{20}{18-i}$$

Problem 7

Proceed as in the derivation of Catalan numbers. The answer is $\binom{m+n}{n} - \binom{m+n}{n-1} = \frac{m-n+1}{m+1}\binom{m+n}{n}$.

Problem 8

(a) 50 objects (people)

take 5 (first five runner position)

$$= P(50,5) = \frac{50!}{45!}$$

(b) For top 3 runners

Krishna & Shyam can finish in $3! = 6$ ways.

For each of these 6 ways, there are $P(48,3)$ ways for other 3 finishers. [For top 5]

Hence, by product rule

Ways to award trophies with Shyam & Krishna among top 3 $= 6 * P(48,3)$

1. How many sorted arrays of size n are there if each element of the array is an integer in the range 1,2,3,...,r?

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Let there be x_i occurences of $i \in \{1,2,3,\dots r\}$

So we have,

$$x_1 + x_2 + x_3 + \dots x_r = n \quad \forall x_i \geq 0$$

Hence,

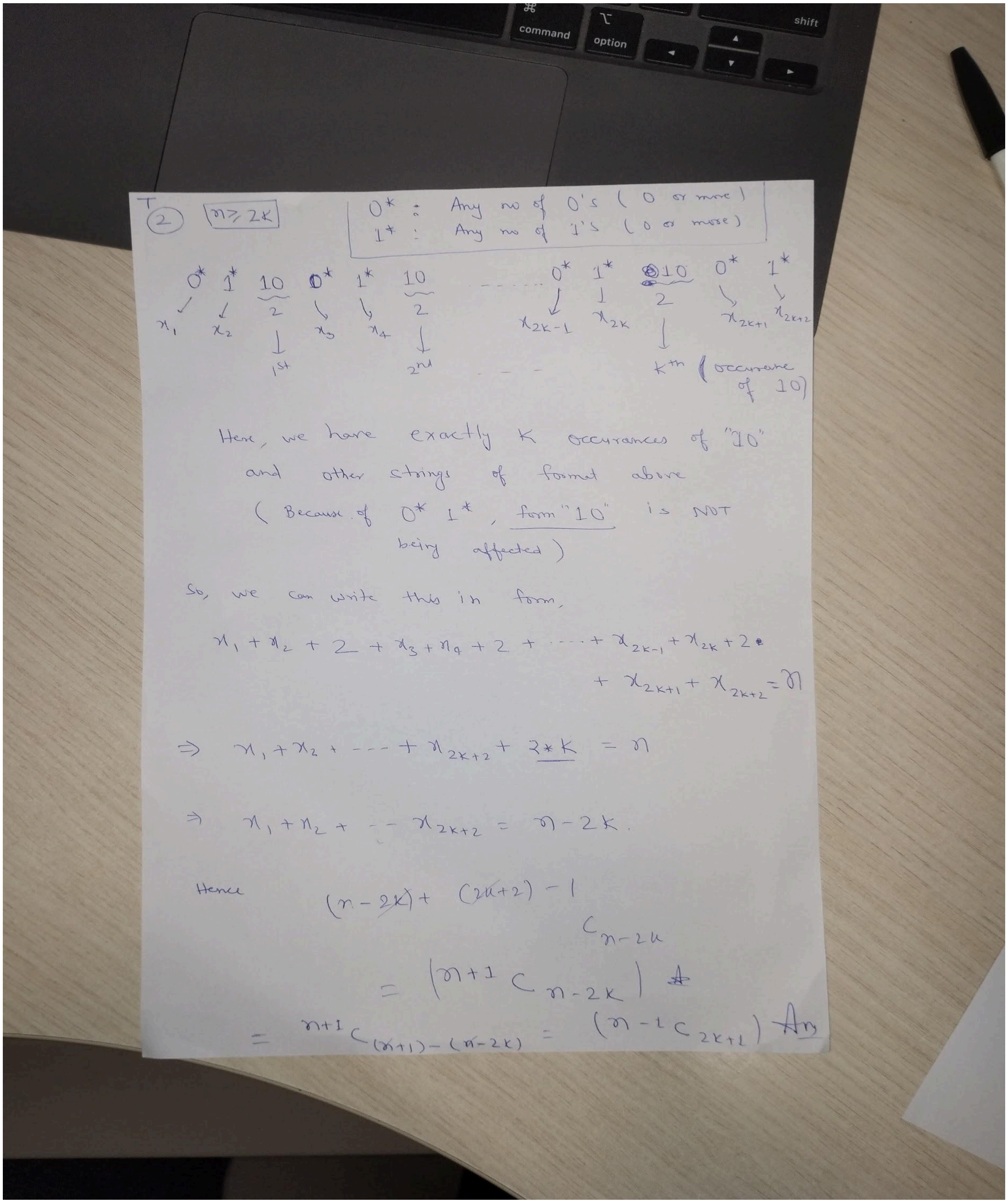
~~According to selecting~~

This matches with selecting r objects with repeatation from n objects s.t. we are selecting 0 or more of same object.

$$\Rightarrow \text{Total no of methods} = \binom{n+r-1}{n}$$

$$= \binom{n+r-1}{r-1} \quad \underline{\text{Ans}}$$

2. How many binary strings of length n are there with exactly k occurrences of the pattern 10? Assume that n >= 2k.



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(2) $\boxed{n \geq 2k}$

$\boxed{\begin{array}{l} 0^* : \text{Any no of 0's (0 or more)} \\ 1^* : \text{Any no of 1's (0 or more)} \end{array}}$

$$\underset{x_1}{0^*}\ \underset{x_2}{1^*}\ \underbrace{10}_{2}\ \underset{x_3}{0^*}\ \underset{x_4}{1^*}\ \underbrace{10}_{2}\ \cdots\cdots\ \underset{x_{2k-1}}{0^*}\ \underset{x_{2k}}{1^*}\ \underbrace{10}_{2}\ \underset{x_{2k+1}}{0^*}\ \underset{x_{2k+2}}{1^*}$$

1st, 2nd, ---, k^{th} (occurrence of 10)

Here, we have exactly k occurrances of "10" and other strings of format above

(Because of $0^*\ 1^*$, form "10" is NOT being affected)

So, we can write this in form,

$$x_1 + x_2 + 2 + x_3 + x_4 + 2 + \cdots + x_{2k-1} + x_{2k} + 2 + x_{2k+1} + x_{2k+2} = n$$

$$\Rightarrow x_1 + x_2 + \cdots + x_{2k+2} + \underline{2 * k} = n$$

$$\Rightarrow x_1 + x_2 + \cdots x_{2k+2} = n - 2k.$$

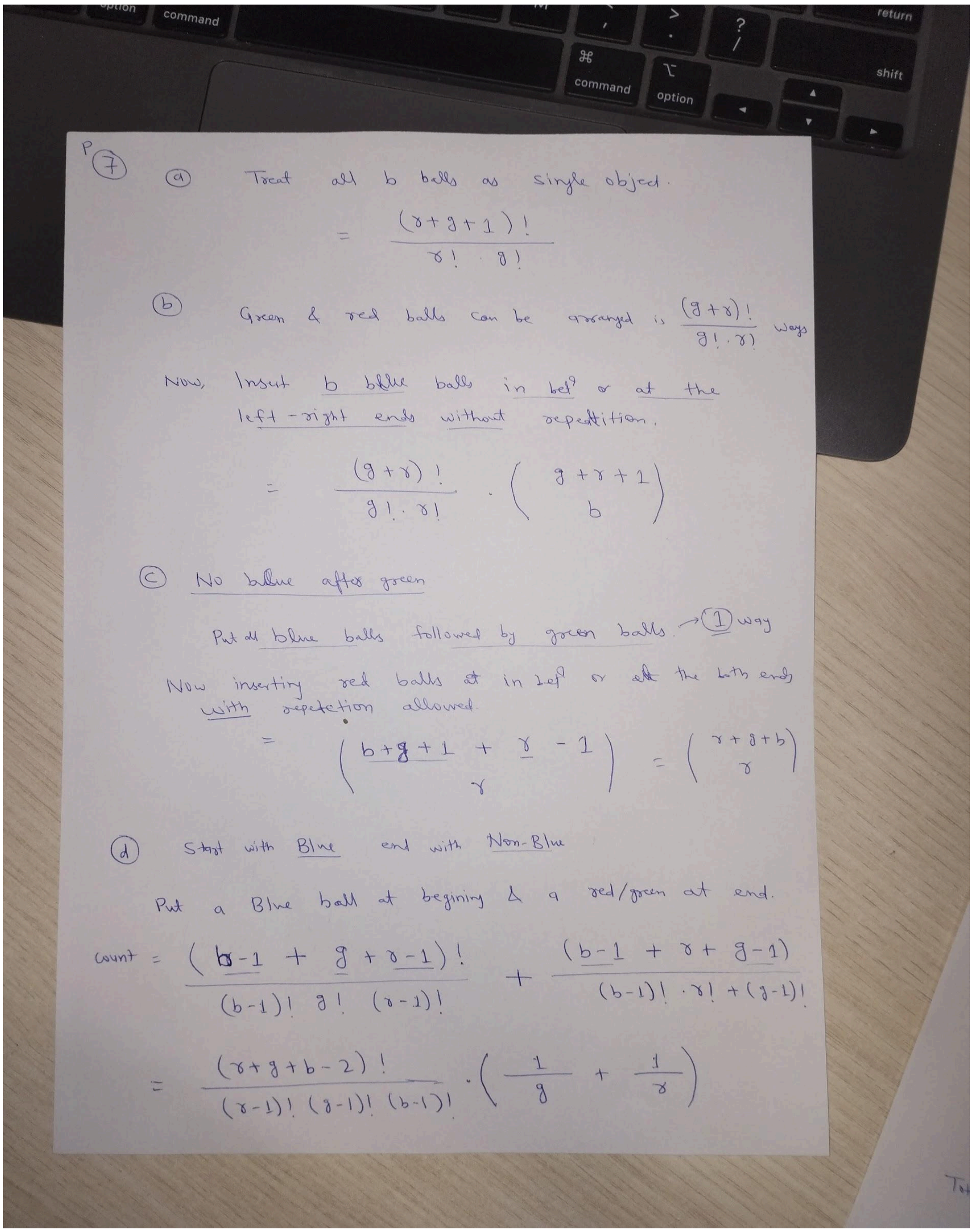
Hence

$$^{(n-2k)+(2k+2)-1}C_{n-2k}$$

$$= \left(^{n+1}C_{n-2k} \right)$$

$$= {}^{n+1}C_{(n+1)-(n-2k)} = \left(^{n+1}C_{2k+1} \right) \text{ Ans}$$

3. You are given r red balls, g green balls, and b blue balls. Assume that r, g, b are positive integers. Your task is to arrange the balls on a line subject to the following conditions. Find the count of all possible arrangements in each case.
(a) All blue balls appear together.
(b) No two blue balls appear together.
(c) No blue ball can appear after any green ball.
(d) The arrangement must start with a blue ball and end with a non-blue ball.



P 7

(a) Treat all b balls as single object.

$$= \frac{(r+g+1)!}{r! \cdot g!}$$

(b) Green & red balls can be arranged is $\frac{(g+r)!}{g! \cdot r!}$ ways

Now, Insert b blue balls in bet^n or at the left-right ends without repetition.

$$= \frac{(g+r)!}{g! \cdot r!} \cdot \binom{g+r+1}{b}$$

(c) No blue after green

Put all blue balls followed by green balls → 1 way

Now inserting red balls at in bet^n or at the both ends with repetition allowed.

$$= \binom{b+g+1+r-1}{r} = \binom{r+g+b}{r}$$

(d) Start with Blue end with Non-Blue

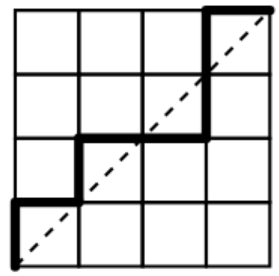
Put a Blue ball at begining & a red/green at end.

$$\text{count} = \frac{(b-1+g+r-1)!}{(b-1)!\ g!\ (r-1)!} + \frac{(b-1+r+g-1)}{(b-1)! \cdot r! + (g-1)!}$$

$$= \frac{(r+g+b-2)!}{(r-1)!\,(g-1)!\,(b-1)!} \cdot \left(\frac{1}{g} + \frac{1}{r}\right)$$

4.

Consider paths in the grid from $(0,0)$ to (n,n) consisting of right and up movements only. Such a path is called diagonal-crossing if it crosses the main diagonal $y = x$ at least once. For example, the adjacent figure shows a path (for $n = 4$) that crosses the diagonal twice (horizontally at $(2,2)$ and vertically at $(3,3)$). Prove that the total number of diagonal-crossing paths from $(0,0)$ to (n,n) is $(n-1)C(n)$, where $C(n)$ is the n-th Catalan number. (**Hint:** First figure out the count of paths that are *not* diagonal-crossing.)



Solution The total number of paths from $(0,0)$ to (n,n) is $\binom{2n}{n}$. The count of paths that never go above the main diagonal is $C(n)$. If we reflect these paths about the line $y = x$, we get all the paths that never go below the main diagonal. All of the remaining paths are diagonal-crossing, so their count is

$$\binom{2n}{n} - 2C(n) = \binom{2n}{n} - \frac{2}{n+1}\binom{2n}{n} = \frac{n-1}{n+1}\binom{2n}{n} = (n-1)C(n).$$

5. The tree data structure is a specialized data structure to store data in a hierarchical manner. It is used to organize and store data in the computer to be used more effectively. The tree data structure has roots, branches, and leaves connected. A full binary tree is a special kind of binary tree that has either zero or two child nodes for each node. What is the total number of full binary trees with n leaves? (Suppose the tree is unlabelled)

Ordered full binary trees with n leaves ↔ Complete binary parenthesizations of n factors i.e Catalan(n-1).

An unlabelled full binary tree with n=4 leaves.