CS21201 Discrete Structures

Tutorial

Proof Techniques, Induction

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1. For all positive integers n, show that there exists a prime > n.
2. Let x be a non-zero real number such that $x + \frac{1}{x}$ is an integer. Prove by induction on n that $x^n + \frac{1}{x^n}$ is an integer for all n ≥ 1.
3. Let n be a positive integer. Consider all non-empty subsets of {1,2,3,...,n}, that do not contain consecutive integers. Let S_n denote the sum of the squares of the products of the elements in these subsets.

   For example, for n = 5, these subsets are
   $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{1, 3, 5\}$

   Therefore S_5 is equal to:
   $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + (1 * 3)^2 + (1 * 4)^2 + (1 * 5)^2 + (2 * 4)^2 + (2 * 5)^2 +$
   $+ (3 * 5)^2 + (1 * 3 * 5)^2 = 719$

   Prove that $S_n = (n + 1)! - 1$ for all $n \geq 1$.

4. Show by induction that $\forall n \in \mathbb{N}$,
   $$f(n) = \sum_{k=0}^{n} \binom{n+k}{k} \frac{1}{2^k} = 2^n$$
5. Are there three consecutive positive integers whose product is a perfect square - that is, do there exist $m, n \in \boldsymbol{Z}^+$ with $m * (m + 1) * (m + 2) = n^2$?
