CS21201 Discrete Structures
Practice Problems
Recurrence Relations

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1. Let $a_n, n \geq 0$ be the count of strings over $\{0, 1, 2\}$ containing no consecutive 1's and no consecutive 2's. Find a recurrence relation for a_n and solve it.
2. Let $a_n, n \geq 1$ satisfy $a_1 = 1$ and
$$a_n = \begin{cases} 2a_{n-1} & \text{if } n \text{ is odd} \\ 2a_{n-1}+1 & \text{if } n \text{ is even} \end{cases} \quad \text{for } n \geqslant 2$$
Develop a recurrence relation for a_n that holds for both odd and even n, and solve it.
3. Solve the following recurrence relation and deduce the closed-form expression for $T(n)$.
$$T(n) = \begin{cases} \sqrt{n}T(\sqrt{n}) + n(\log_2 n)^d, & \text{if } n > 2 \\ 2 & , \text{ if } n = 2 \end{cases} \quad (d \geqslant 0)$$
4. Deduce the running times of divide-and-conquer algorithms in the big-Θ notation if their running times satisfy the following relations:
    a. $T(n) = T(2n/3) + T(n/3) + n \log n$
    b. $T(n) = T(n/5) + T(7n/10) + n$

5. Pell numbers are defined as $P_0 = 0,\ P_1 = 2,\ P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$.
    a. Deduce a closed-form formula for P_n.
    b. Prove that $\begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n$ for all $n \geq 1$.
    c. Prove that $\lim_{n\to\infty} \frac{P_{n-1}+P_n}{P_n} = \sqrt{2}$.
    d. Prove that if P_n is prime, then n is also prime.

6. Consider a non-homogeneous recurrence of the form:
$$a_n = c_{k-1}a_{n-1} + c_{k-2}a_{n-2} + \ldots + c_0 a_{n-k} + p_1(n)s_1^n + p_2(n)s_2^n.$$
Here, $c_{k-1}, c_{k-2}, \ldots, c_0$ are constants (with $c_0 \neq 0$), $p_1(n)$ and $p_2(n)$ are non-zero polynomials in n, and s_1, s_2 are distinct non-zero constants. Propose a method to solve this recurrence.

7. Let $\{a_n\}$ be a sequence such that $a_1 = 1,\ a_{n+1} = \frac{1}{16}(1 + 4a_n + \sqrt{1 + 24a_n}),\ n \geq 1$. Find a_n.

8. Let $a_1 = 1,\ a_n = \sum_{k=1}^{n-1} (n-k)a_k,\ \forall n \geq 2$. Find a_n.

9. $a_0 = 0, a_1 = 1, a_n = 2a_{n-1} + a_{n-2}, n \geq 2$. Prove that $2^k | a_n$ if and only if $2^k | n$.
