# CS21201 Discrete Structures
## Practice Problems
## Generating Functions

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1. Find the generating function of the sequence $1, 2, 0, 3, 4, 0, 5, 6, 0, 7, 8, 0, ....$

2. Let the two-variable sequence $a_{m,n}$ be recursively defined as follows.

$$a_{m,n} = \begin{cases} 1 & \text{if } m = 0 \text{ or } n = 0, \\ a_{m-1,n} + a_{m,n-1} & \text{if } m \geqslant 1 \text{ and } n \geqslant 1. \end{cases}$$

   Find the generating function $A(x, y) = \sum_{m,n\geq 0} a_{m,n} x^m y^n$. From this, derive a closed-form formula for $a_{m,n}$.

3. Prove that the generating function for Catalan numbers is $f(x) = \frac{1-\sqrt{1-4x}}{2x}$.

4. Let $a_n$, $n \geq 0$, be the sequence satisfying

$$a_0 = 1$$
$$a_n = 2 + 2a_0 + 2a_1 + 2a_2 + ... + 2a_{n-2} + a_{n-1} \text{ for } n \geq 1$$

   Deduce that the generating function of this sequence is $\frac{1+x}{1-2x-x^2}$. Solve for $a_n$.

5. Let $m \geqslant 1$ be an integer constant. Let $b_n^{(m)}$ denote the number of ordered partitions (that is, compositions) of the integer $n \geq 0$ such that no summand is larger than $m$.
   (a) Prove that the (ordinary) generating function of $b_n^{(m)}$ is

$$B^{(m)}(x) = \frac{1-x}{1-2x+x^{m+1}}$$

   (b) From the formula of Part (a), deduce that $b_n^2 = F_{n+1}$, where $F_0, F_1, F_2, ....$ is the sequence of Fibonacci numbers.

6. How many bit strings of length n are there in which 1's always occur in contiguous pairs? You should consider strings of the form 0011011110, but not of the form 0110111110, because the last 1 is not paired.

7. Let $a_n = \sum_{i=n}^{\infty} \frac{2^i}{i!}$ for all integers $n \geq 0$.
   (a) Find a closed-form expression for the (ordinary) generating function $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + .... + a_nx^n + ...$ of the sequence $a_0, a_1, a_2, ... a_n, ...$

   (b) Use the expression for $A(x)$ in Part (a) to prove that $\sum_{n=0}^{\infty} a_n = 3e^2$.

8. Let $u, v, s, t$ be positive constant values. Consider the sequence $a_0, a_1, a_2, \ldots$. defined recursively as follows.

$$a_0 = u$$

$$a_1 = v$$

$$a_n = sa_{n-1} + ta_{n-2} \text{ for all } n \geq 2.$$

For $n \geq 3$, we have

$$a_n = sa_{n-1} + ta_{n-2} = s(sa_{n-2} + ta_{n-3}) + ta_{n-2} = (s^2 + t)a_{n-2} + sta_{n-3}$$

In view of this, consider the sequence $b_0, b_1, b_2, \ldots$ defined recursively as follows:

$$b_0 = u$$

$$b_1 = v$$

$$b_n = (s^2 + t)b_{n-2} + stb_{n-3} \text{ for all } n \geq 3.$$

Demonstrate that the generating functions of both the sequences are the same.