Discrete Structures 2024

Sets, Relation, Function - Practice Problems

September 11, 2024

1. Let A, B, $C \subseteq U$. Prove that $(A - B) \subseteq C$ if and only if $(A - C) \subseteq B$.

2. Let $A, B \subseteq \mathbb{R}$, where

$$A = \{x \mid x^2 - 7x < -12\} \quad \text{and} \quad B = \{x \mid x^2 - x < 6\}.$$

   Determine $A \cup B$ and $A \cap B$.

3. Define a relation ρ on $\mathbb{N}$ as $a \,\rho\, b$ if and only if a has the same set of prime divisors as b. For example, 5 is related to $25 = 5^2$, $12 = 2^2 \times 3$ is related to $54 = 2 \times 3^3$, but 12 is not related to $16 = 2^4$, nor to $180 = 2^2 \times 3^2 \times 5$.
   (a) Prove that ρ is an equivalence relation on $\mathbb{N}$.
   (b) Find a unique representative from each equivalence class of ρ.

4. Let $f : \mathbb{Z} \to \mathbb{N}$ be defined by

$$f(x) = 2x - 1 \quad \text{if } x > 0, \quad \text{and} \quad f(x) = -2x \quad \text{for } x < 0.$$

   (a) Prove that f is one-to-one and onto.
   (b) Determine f^{-1}.

5. In ten days, Ms. Rosatone typed 84 letters to different clients. She typed 12 of these letters on the first day, seven on the second day, and three on the ninth day, and she finished the last eight on the tenth day. Show that for three consecutive days, Ms. Rosatone typed at least 25 letters.

6. Suppose you set your computer password of length m from a fixed chosen set of n different characters available in the keyboard ($m \geq n$). How many different passwords can you set so that at least one occurrence of each symbol (from the n chosen set of keyboard symbols) will be present?

7. For $m, n \in \mathbb{Z}^+$ with $m < n$, prove that,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m = 0.$$

8. Let $A = \mathbb{R} \times \mathbb{R}$ be the set of all ordered pairs of real numbers. Define a binary operation $\diamond$ on A as follows:

   For $(a_1, b_1), (a_2, b_2) \in A$,

$$(a_1, b_1) \diamond (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + b_1 a_2)$$

   (a) Prove or disprove that $\diamond$ is commutative.
   (b) Prove or disprove that $\diamond$ is associative.

(c) Find the identity element for $\diamond$, if it exists.

(d) For any $(a, b) \in A$ where $a^2 + b^2 \neq 0$, find the inverse element under $\diamond$.

9. Let $S = \mathbb{Z}^+ \times \mathbb{Z}^+$. Define a relation R_1 on S as, $(x, y)R_1(m, n)$ if and only if $x \leq m$ and $y \leq n$. Prove or disprove that R_1 is a total-order on S.

10. A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken a course in at least one of Spanish French and Russian, how many students have taken a course in all 3 languages.

11. Professor Bailey has just completed writing the final examination for his course in advanced engineering mathematics. This examination has 12 questions, whose total value is to be 200 points. In how many ways can Professor Bailey assign the 200 points if (a) each question must count for at least 10, but no more than 25, points? (b) each question must count for at least 10, but not more than 25, points and the point value for each question is to be a multiple of 5?