Discrete Structures 2024

Pigeonhole Principle - Practice Problems

September 3, 2024

1. You pick six points in a 3×4 rectangle. Prove that two of these points must be at a distance $\leq \sqrt{5}$.

2. Let p be a prime number, and x an integer not divisible by p. Prove that there exist non-zero integers a and b of absolute values less than $\sqrt{p}$ such that p divides $ax - b$.

3. Let $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$. Use the pigeonhole principle to prove that $ua + vb = 1$ for some $u, v \in \mathbb{Z}$.

4. Let p be a prime number, and x an integer not divisible by p. Now assume that p is of the form $4k + 1$. We know from number theory that in this case, there exists an integer x such that p divides $x^2 + 1$. Show that $p = a^2 + b^2$ for some integers a, b.

5. Let $n \geq 10$ be an integer. You choose n distinct elements from the set $\{1, 2, 3, \ldots, n^2\}$. Prove that there must exist two disjoint subsets of the chosen numbers whose sums are equal.

6. Let $p(x)$ be a polynomial with integer coefficients, having three distinct integer roots a, b, c. Prove that the polynomials $p(x) \pm 1$ cannot have any integer roots.

7. 65 distinct integers are chosen in the range $1, 2, 3, \ldots, 2022$. Prove that there must exist four of the chosen integers (call them a, b, c, d) such that $a - b + c - d$ is a multiple of 2022.