CS21201 Discrete Structures

Practice Problems

Recursive Constructions, Loop Invariance

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1. Let $s(n, m)$ denote the number of permutations of $1, 2, 3, ..., n$ that have exactly m cycles. For example, the permutation $3, 1, 6, 8, 2, 5, 7, 4$ $(for\ n = 8)$ has three cycles $(1, 3, 6, 5, 2), (4, 8), (7)$. The numbers $s(n, m)$ are called Stirling numbers of the first kind. Prove that $s(m, n) = s(m - 1, n - 1) + (m - 1)s(m - 1, n)$.

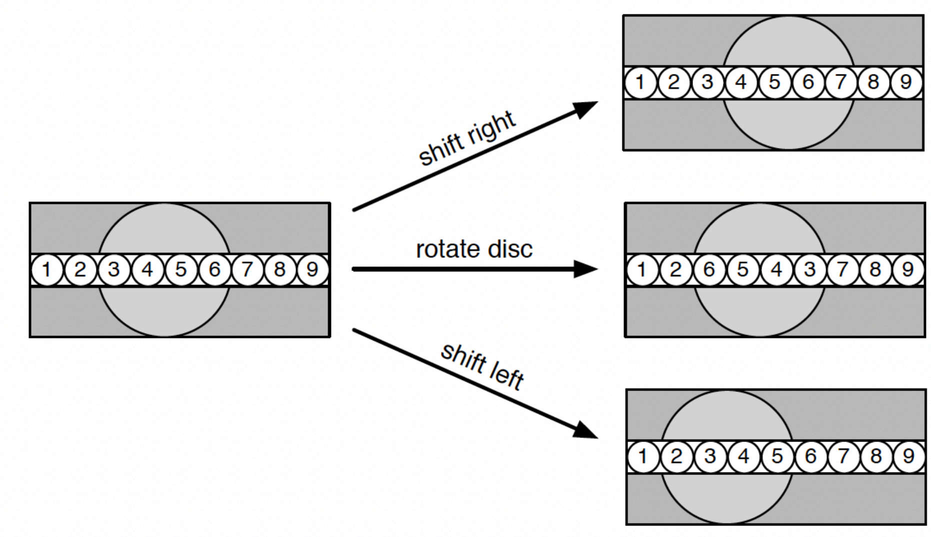
2. Prove using the theory of loop invariance that the following function prints d, u, v, where d = gcd(x, y) = ux + vy with d ∈ N and u, v ∈ Z. Assume that both the arguments x and y are supplied with positive values.

```
void egcd ( unsigned int x, unsigned int y ) {
      int a1, a2, u1, u2;
      a1 = x; a2 = y; u1 = 1; u2 = 0;
      while (a1 != a2) {
            if (a1 > a2) { a1 = a1 - a2; u1 = u1 - u2; }
            else { a2 = a2 - a1; u2 = u2 - u1; }
      }
      printf("%d, %d, %d\n", a1, u1, (a1 - u1 * x) / y);
}
```

3. Consider a track holding 9 circular tiles. In the middle is a disc that can slide left / slide right and rotate 180◦. Thus, there are three ways to manipulate the current state:

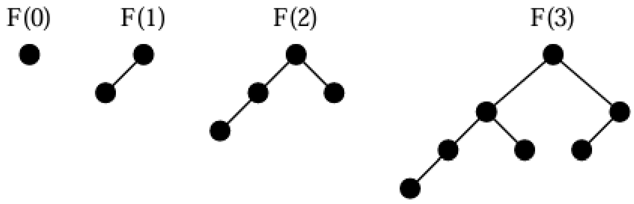
   Shift Right: The center disc is moved one unit to the right (if there is space)
   Rotate Disc: The four tiles in the center disc are reversed
   Shift Left: The center disc is moved one unit to the left (if there is space)

   The figure below illustrates these three manipulations.



   Prove that if the track starts in an initial state with all but tiles 1 and 2 in their natural order, it is impossible to reach the goal state where all the tiles are in their natural order. These states are shown below.

4. Fibonacci trees F(n) are recursively defined for n ⩾ 0 as follows. F(0) is a single node. F(1) has two nodes with the root having only a left child. For n ⩾ 2, the tree F(n) consists of a root. Its left subtree is $F(n-1)$, and its right subtree is $F(n-2)$. The following figure shows the first four Fibonacci trees.



Prove the following assertions for all n ⩾ 0.
(a) F(n) contains $F_{n+3} - 1$ nodes (where Fi is the i-th Fibonacci number).
(b) F(n) contains F_{n+1} leaf nodes.
(c) The height of F(n) is n.
(d) At every non-leaf node v of F(n), the height of the left subtree of v is one more than the height of the right subtree of v. (Note that the empty tree has height −1, and a single-node tree has height 0.)

5. Give a recursive definition of each of these sets of ordered pairs of positive integers.Use structural induction to prove that the recursive definition you found is correct. [Hint: To find a recursive definition, plot the points in the set in the plane and look for patterns.]
   a. S = {(a, b) | a ∈ Z^+, b ∈ Z^+, and a + b is even}
   b. S = {(a, b) | a ∈ Z^+, b ∈ Z^+, and a or b is odd}
   c. S = {(a, b) | a ∈ Z^+, b ∈ Z^+, and a+b is odd, 3 | b}

6. Let S be a set defined recursively by
Basis step: 1 ∈ S.
Recursive step: For a positive integer k, if any element of the set {k, 2k+1, 3k} belongs to S, then all belong to S.
For example, if 6 ∈ S, then 2 ∈ S, 5 ∈ S, 13 ∈ S and 18 ∈ S. Does 2023 ∈ S?

7. The positive integers 1,2,...,n are arranged in a random order. In one operation, two integers are interchanged. Prove that the initial ordering can never be reached after an odd number of operations.

8. Give a recursive algorithm for tiling a $2^n x\, 2^n$ checker board with one square missing using right triominoes. Right triominoes are L-shaped tiles made up of three squares that cover exactly 1 unit square of the board.