CS21201 Discrete Structures
Practice Problems
Proof Techniques, Induction

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1. Prove that every positive integer greater than one can be factored as a product of primes. [Hint: Prove this using well-ordering theorem]

2. Prove that every positive integer can be written as a product of prime factors, and this product is unique up to the reordering of factors (also known as the Fundamental Theorem of Arithmetic). [Hint: Prove this using Principle of Mathematical Induction]

3. Prove that √n is irrational if and only if n is not a perfect square.

4. Using mathematical induction, prove that $2^n < n! < 2^{n\log_2 n}$, $\forall\, n >= 4$.

5. Let a,b be two positive integers, and d = gcd(a,b) = ua + vb with u, v ∈ ℤ. Prove that u and v can be so chosen that $|u| < \frac{b}{d}$ and $|v| \le \frac{a}{d}$.

6. You have coins of two integral denominations a,b > 1 with gcd(a,b) = 1. Prove that any integer amount n ≥ (a−1)(b−1) can be changed by coins of these two denominations. $[\exists x, y > 0,\ n = xa + yb]$

7. Let a,b be as in the last question. Prove that the amount (a−1)(b−1)−1 cannot be changed by coins of denominations a and b.

8. Let F_n denote the n-th Fibonacci number.
    a. Prove that for all integers m,n with m ≥ 1 and n ≥ 0, we have $F_{m+n} = F_m F_{n+1} + F_{m-1} F_n$.
    b. Let m,n ∈ ℕ. Prove that if m|n, then $F_m | F_n$.
    c. What about the converse of Part (b)?
    d. Prove $gcd(F_m, F_n) = F_{gcd(m,n)} \forall m, n \ge 1$.

9. Using the principle of mathematical induction, prove the following statements.
    a. $\forall n \ge 4$, the nth-Catalan number satisfies $C_n \le 2^{2n-4}$.
    b. The harmonic numbers $H_n = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}$ satisfy $ln(n + 1) \le H_n \le ln\, n + 1, \forall n \ge 1$.
