CS21201/CS21001 Discrete Structures, Autumn 2024–2025

Class Test 2

8–November–2024 | 06:30pm–07:30pm | Maximum marks: 30

Roll no: ________________ **Name:** ____

[*Write your answers in the question paper itself. Be brief and precise. Answer all questions.*]

1. For each of the following pairs, mention clearly whether or not they have the same size, with brief justification. **(4)**

Solution For each of these, 0.5 marks if the answer is correct, 1 mark if explanation is correct (even if partially)

- (Set of integers, Set of rational numbers)

Solution They have the same size. Both are countably infinite sets (and size is $\aleph_0$)

- (Set of real numbers, all subsets of natural numbers)

Solution They have the same size. Both are uncountable (with size as c or $2^{\aleph_0}$)

- (Set of all polynomials with integer coefficients, infinite length bit strings)

Solution They do not have the same size. While the set of all polynomials with integer coefficients is countable, number of infinite length bit strings is uncountable.

- (all subsets of natural numbers, all finite-length subsets of rational numbers)

Solution They do not have the same size. While all subsets of natural numbers are uncountable, all finite-length subsets of rational numbers is countable.

2. Suppose you have a finite alphabet Σ of length k. Prove that Σ^* (set of all finite-length strings over Σ) is countable by giving an explicit bijection from natural numbers to Σ^*. What about the set of all languages over Σ^*, where a language is defined as a subset of all the strings? Briefly justify your answer. **(6)**

Solution Σ^* is countable. Σ^0 contains only empty string (1), Σ^1 contains strings of length 1 (k), Σ^2 contains strings of length 2 (k^2), and so on.

To get an explicit bijection from natural numbers, for a given number n, consider the largest y such that $k^0 + k^1 + k^2 + \ldots + k^y = \frac{(k^y-1)}{(k-1)} < n$. So, this will map to a string of length $y+1$. You can now give this explicit mapping – there are k^{y+1} strings of length $y+1$. Put them in lexicographical order. n will map to $(n - \frac{(k^y-1)}{(k-1)})$th string. (4 marks, give partial marks if the mapping is not clear)

The set of all languages over Σ^* is the power set of Σ^*. Since Σ^* is an infinite set, this would be an uncountable set. (2 marks)

3. Show that the set of real numbers R have the same size as the set of negative real numbers R^-. **(5)**

Solution as $|R^-| \leq |R|$, we can show this by giving an injective map from R to R^- (thus $|R| \leq |R^-|$).

Consider the following mapping $f : R \rightarrow R^-$

$$f(x) = -\frac{x}{x+1}, x \in R^+$$

$$f(x) = -[\frac{-x}{-x+1} + 1], x \in R^-$$

$$f(x) = -2, x = 0$$

4. A GOOD codeword is defined as an n-digit number in decimal notation with even number of 0s. Let d_n denotes the number of GOOD codewords of length n. The sequence $\{d_1, \cdots, d_n, \cdots\}$ satisfies the following recurrence relation:

$$d_n = 8d_{n-1} + 10^{n-1},\ d_1 = 9.$$

Use generating functions to find an explicit formula for d_n. **(9)**

Solution Example 17 (Page 573) of Discrete Mathematics and its Applications (Eighth Edition) by Kenneth H. Rosen.

Instruction: If GF is not used to solve the problem, deduct 50% marks and evaluate.

5. In how many ways 28 chocolates can be distributed among four kids so that each one of them gets at least 2 chocolates, but no more than 9 chocolates? Use generating functions to solve this problem. **(6)**

Solution [Hint] Find coefficient of x^{28} in the generating function $(x^2+x^3+\cdots+x^9)^4 = x^8(1+x+\cdots+x^7)^4 = x^8(\frac{1-x^8}{1-x})^4$. Thus coefficient of x^{20} in $(\frac{1-x^8}{1-x})^4$ gives the solution. Expand $(1-x^8)^4$ and $(1-x)^{-4}$, find coefficient of x^{20} and check whether the solution is 161.