CS21201/CS21001 Discrete Structures, Autumn 2024–2025

Class Test 1

30–August–2024 | 06:30pm–07:30pm | Maximum marks: 30

Roll no: ____________________ **Name:** ____

[*Write your answers in the question paper itself. Be brief and precise. Answer all questions.*]

1. Count the solutions for the following. **(7)**

(a) $x_1 + x_2 + x_3 + x_4 = 50$ with integers $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$.

Solution This can be transformed to $y_1 + y_2 + y_3 + y_4 = 40$ with $y_i \geq 0$. The number of solutions are $\binom{43}{40}$. [3 marks]

(b) $x_1 + x_2 + x_3 + x_4 \geq 50$ with integers $x_1 \leq 11$, $x_2 \leq 22$, $x_3 \leq 33$, $x_4 \leq 44$.

Solution Consider $y_1 = 11 - x_1$, $y_2 = 22 - x_2$, $y_3 = 33 - x_3$. $y_4 = 44 - x_4$.

The equation then becomes $y_1 + y_2 + y_3 + y_4 \leq 60$ with $y_i \geq 0$.

We can introduce $y_5 = 60 - (y_1 + y_2 + y_3 + y_4)$, $y_5 \geq 0$.

So, the equation becomes $y_1 + y_2 + y_3 + y_4 + y_5 = 60$

The number of solutions are $\binom{64}{60}$. [4 marks]

2. Answer the following questions. **(8)**

(a) Let $P(x)$ and $Q(x)$ be predicates involving an integer valued variable x. Prove or disprove: $\forall x[P(x) \Rightarrow Q(x)]$ is logically equivalent to $\forall x[P(x)] \Rightarrow \forall x[Q(x)]$

Solution You can disprove this by providing a counter-example.

Let $P(x)$: "x is even", and $Q(x)$: "x is divisible by 4". Clearly, $\forall x[P(x) \Rightarrow Q(x)]$ is false. $\forall x[P(x)]$ is also false, but for the implication $p \Rightarrow q$, if p is false, the implication is true. Hence, $\forall x[P(x)] \Rightarrow \forall x[Q(x)]$ is true. [4 marks]

(b) Let p, q and r be the following propositions:

p: It is raining

q: I have a headache

r: I attend the lecture

Use Propositional logic to express the statement, "If it is not raining and I do not have a headache, then I attend the lecture" using the propositions defined above. Find its negation, as well as converse, and express those in English.

Solution $P = (\neg p \wedge \neg q) \rightarrow r$

$P \equiv \neg(\neg p \wedge \neg q) \vee r \equiv (p \vee q \vee r)$, and so $\neg P \equiv (\neg p \wedge \neg q \wedge \neg r)$

The negation may thus be rendered in English as: "It is not raining and I do not have a headache and I do not attend the lecture."

The converse of P is $r \rightarrow (\neg p \wedge \neg q)$ and is rendered in English as, "If I attend the lecture, then it is not raining and I do not have a headache." [1 mark for proposition, 1 mark for negation, 1 mark for converse, 1 mark for expressing in English.]

3. Write 1998 numbers $\{1,2,\cdots,1998\}$ on a green board. Pick any two numbers m and n from the numbers written on the board. Erase m, n and write $|m-n|$ on the board. Continue this process until only one integer is written on the board. Prove that this integer must be odd. **(9)**

Solution The parity of the sum of the numbers written on the board never changes, because $m+n$ and $|m-n|$ have the same parity (and at each step we reduce the sum by $m+n$ but increase it by $|m-n|$). Thus the number at the end of the process must have the same parity as $1+2+\cdots+2\cdot 99 = 19701$, which is odd.

4. Using the principle of mathematical induction prove that:

$$(n+1)(n+2)\cdots 2n = 2^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1),\ \forall n \in \mathbb{N}$$

(6)

Solution [Hint:] Say $f(n) = (n+1)(n+2)\cdots 2n$ and $g(n) = 2^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)$.

Basis: Show $f(1) = g(1)$.

Say for some $n \in \mathbb{N}$, $f(n) = g(n)$. Show that $f(n+1) = f(n)(4n+2)$ and $g(n+1) = g(n)(4n+2)$. Then

$$\frac{f(n+1)}{f(n)} = \frac{g(n+1)}{g(n)} \Rightarrow f(n+1) = g(n+1)\cdot$$

Now use PMI to prove the statement.