Reasoning with Bayes Networks

Course: CS40022
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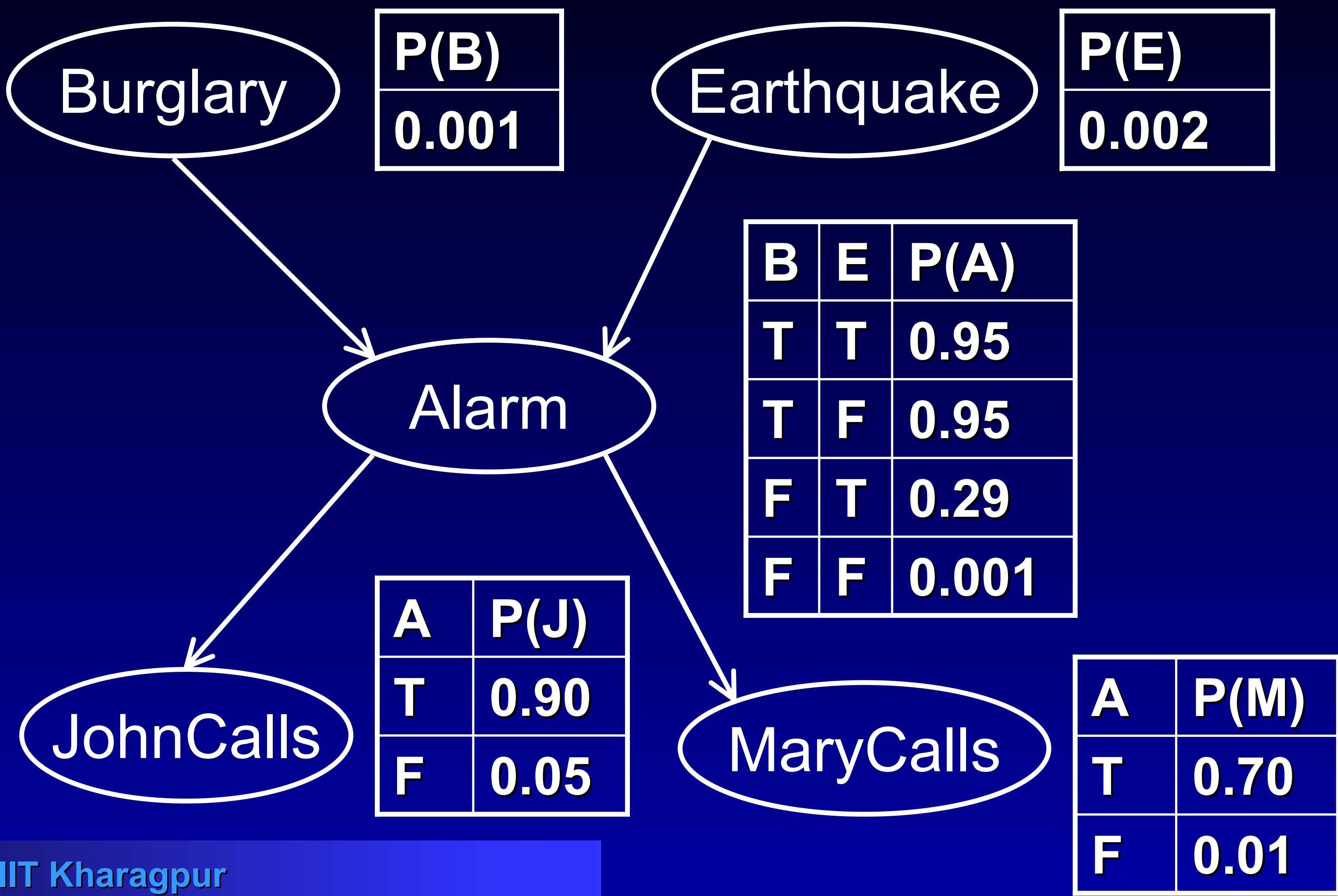
Belief Network Example



P(B)
0.001

P(E)
0.002

B	E	P(A)
T	T	0.95
T	F	0.95
F	T	0.29
F	F	0.001

A	P(J)
T	0.90
F	0.05

A	P(M)
T	0.70
F	0.01

Conditional independence

$$P(x_1,\ldots,x_n)$$
$$= P(x_n \mid x_{n-1},\ldots,x_1)P(x_{n-1},\ldots,x_1)$$
$$= P(x_n \mid x_{n-1},\ldots,x_1)P(x_{n-1} \mid x_{n-2},\ldots,x_1)$$
$$\ldots P(x_2 \mid x_1)P(x_1)$$
$$= \prod_{i=1}^{n} P(x_i \mid x_{i-1},\ldots,x_1)$$

- The belief network represents conditional independence:

$$P(X_i \mid X_i,\ldots,X_1) = P(X_i \mid \mathrm{Parents}(X_i))$$

Inferences using belief networks

- Diagnostic inferences (from effects to causes)
  - Given that JohnCalls, infer that

    P(Burglary | JohnCalls) = 0.016

- Causal inferences (from causes to effects)
  - Given Burglary, infer that

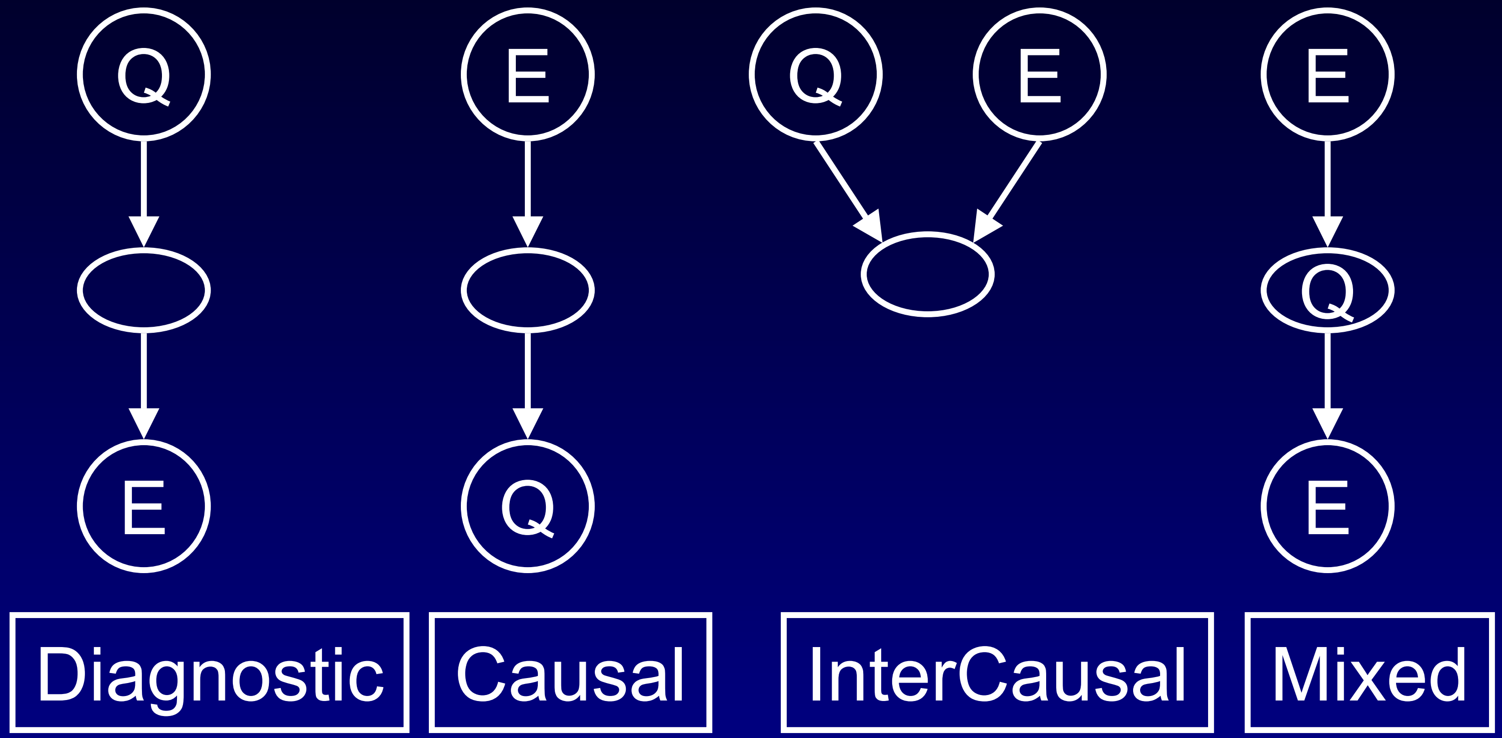
    P(JohnCalls | Burglary) = 0.86 and

    P(MaryCalls | Burglary) = 0.67

Inferences using belief networks

- Intercausal inferences (between causes of a common effect)
  - Given Alarm, we have
    $P(Burglary \mid Alarm) = 0.376.$
  - If we add evidence that Earthquake is true, then $P(Burglary \mid Alarm \wedge Earthquake)$ goes down to 0.003
- Mixed inferences
  - Setting the effect JohnCalls to true and the cause Earthquake to false gives $P(Alarm \mid JohnCalls \wedge \neg Earthquake) = 0.003$

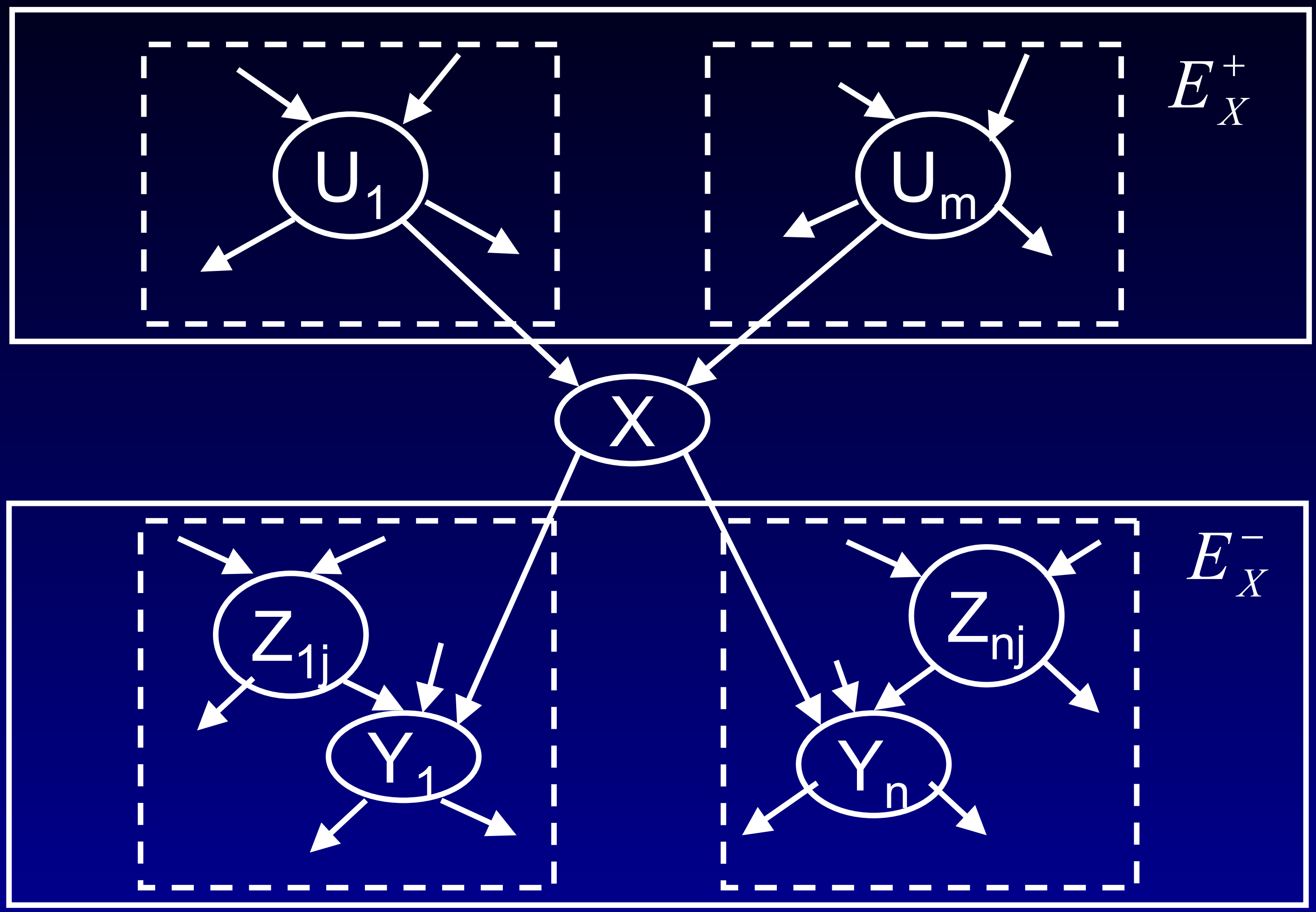
The four patterns



Answering queries

- We consider cases where the belief network is a poly-tree
  - There is at most one undirected path between any two nodes

Answering queries



Answering queries

- $U = U_1 \ldots U_m$ are parents of node X
- $Y = Y_1 \ldots Y_n$ are children of node X
- X is the query variable
- E is a set of evidence variables
- The aim is to compute $P(X \mid E)$

Definitions

- E_X^+ is the causal support for X
  - The evidence variables "above" X that are connected to X through its parents
- E_X^- is the evidential support for X
  - The evidence variables "below" X that are connected to X through its children
- $E_{Ui \setminus X}$ refers to all the evidence connected to node U_i except via the path from X
- $E_{Yi \setminus X}^+$ refers to all the evidence connected to node Y_i through its parents for X

The computation of P(X|E)

$$P(X \mid E) = P(X \mid E_X^-, E_X^+)$$

$$= \frac{P(E_X^- \mid X, E_X^+)\, P(X \mid E_X^+)}{P(E_X^- \mid E_X^+)}$$

- Since X d-separates E_X^+ from E_X^-, we can use conditional independence to simplify the first term in the numerator
- We can treat the denominator as a constant

$$P(X \mid E) = \alpha\, P(E_X^- \mid X) P(X \mid E_X^+)$$

The computation of $P(X \mid E_X^+)$

We consider all possible configurations of the parents of X and how likely they are given E_X^+.

Let U be the vector of parents $U_1, \ldots, U_m$, and let u be an assignment of values to them.

$$P(X \mid E_X^+) = \sum_u P(X \mid u, E_X^+) P(u \mid E_X^+)$$

The computation of $P(X \mid E_X^+)$

$$P(X \mid E_X^+) = \sum_u P(X \mid u, E_X^+) P(u \mid E_X^+)$$

u d-separates X from E_X^+, so the first term simplifies to $P(X \mid u)$

We can simplify the second term by noting

- E_X^+ d-separates each U_i from the others,
- the probability of a conjunction of independent variables is equal to the product of their individual probabilities

$$P(X \mid E_X^+) = \sum_u P(X \mid u) \prod_i P(u_i \mid E_X^+)$$

The computation of $P(X \mid E_X^+)$

$$P(X \mid E_X^+) = \sum_u P(X \mid u) \prod_i P(u_i \mid E_X^+)$$

The last term can be simplified by partitioning E_X^+ into $E_{U1\backslash X}, \ldots, E_{Um\backslash X}$ and noting that $E_{Ui\backslash X}$ d-separates U_i from all the other evidence in E_X^+

$$P(X \mid E_X^+) = \sum_u P(X \mid u) \prod_i P(u_i \mid E_{Ui\backslash X})$$

- $P(X \mid u)$ is a lookup in the cond prob table of X
- $P(u_i \mid E_{Ui\backslash X})$ is a recursive (smaller) sub-problem

The computation of $P(E_X^- \mid X)$

Let Z_i be the parents of Y_i other than X, and let z_i be an assignment of values to the parents

– The evidence in each Y_i box is conditionally independent of the others given X

$$P(E_X^- \mid X) = \prod_i P(E_{Y_i \backslash X} \mid X)$$

The computation of $P(E_X^- \mid X)$

$$P(E_X^- \mid X) = \prod_i P(E_{Yi \backslash X} \mid X)$$

Averaging over Y_i and z_i yields:

$$P(E_X^- \mid X) = \prod_i \sum_{y_i} \sum_{z_i} P(E_{Yi \backslash X} \mid X, y_i, z_i) P(y_i, z_i \mid X)$$

The computation of $P(E_X^- \mid X)$

$$P(E_X^- \mid X) = \prod_i \sum_{y_i} \sum_{z_i} P(E_{Yi \backslash X} \mid X, y_i, z_i) P(y_i, z_i \mid X)$$

Breaking $E_{Yi\backslash X}$ into the two independent components E_{Yi}^- and $E_{Yi\backslash X}^+$

$$P(E_X^- \mid X) = \prod_i \sum_{y_i} \sum_{z_i} P(E_{Yi}^- \mid X, y_i, z_i)$$

$$P(E_{Yi \backslash X}^+ \mid X, y_i, z_i) P(y_i, z_i \mid X)$$

The computation of $P(E_X^- \mid X)$

$$P(E_X^- \mid X) = \prod_i \sum_{y_i} \sum_{z_i} P(E_{Yi}^- \mid X, y_i, z_i)$$

$$P(E_{Yi\backslash X}^+ \mid X, y_i, z_i) P(y_i, z_i \mid X)$$

E_{Yi}^- is independent of X and z_i given y_i, and $E_{Yi\backslash X}^+$ is independent of X and y_i

$$P(E_X^- \mid X) = \prod_i \sum_{y_i} P(E_{Yi}^- \mid y_i) \sum_{z_i} P(E_{Yi\backslash X}^+ \mid z_i) P(y_i, z_i \mid X)$$

The computation of $P(E_X^- \mid X)$

$$P(E_X^- \mid X) = \prod_i \sum_{y_i} P(E_{Yi}^- \mid y_i) \sum_{z_i} P(E_{Yi\backslash X}^+ \mid z_i) P(y_i, z_i \mid X)$$

Apply Bayes' rule to $P(E_{Yi\backslash X}^+ \mid z_i)$:

$$P(E_X^- \mid X) =$$

$$\prod_i \sum_{y_i} P(E_{Yi}^- \mid y_i) \sum_{z_i} \frac{P(z_i \mid E_{Yi\backslash X}^+) P(E_{Yi\backslash X}^+)}{P(z_i)} P(y_i, z_i \mid X)$$

The computation of $P(E_X^- \mid X)$

$$P(E_X^- \mid X) =$$

$$\prod_i \sum_{y_i} P(E_{Y_i}^- \mid y_i) \sum_{z_i} \frac{P(z_i \mid E_{Y_i \backslash X}^+) P(E_{Y_i \backslash X}^+)}{P(z_i)} P(y_i, z_i \mid X)$$

- Rewriting the conjunction of Y_i and z_i:

$$P(E_X^- \mid X) = \prod_i \sum_{y_i} P(E_{Y_i}^- \mid y_i)$$

$$\sum_{z_i} \frac{P(z_i \mid E_{Y_i \backslash X}^+) P(E_{Y_i \backslash X}^+)}{P(z_i)} P(y_i \mid X, z_i) P(z_i \mid X)$$

The computation of $P(E_X^- \mid X)$

$$P(E_X^- \mid X) = \prod_i \sum_{y_i} P(E_{Yi}^- \mid y_i)$$

$$\sum_{z_i} \frac{P(z_i \mid E_{Yi\backslash X}^+) P(E_{Yi\backslash X}^+)}{P(z_i)} P(y_i \mid X, z_i) P(z_i \mid X)$$

$P(z_i \mid X) = P(z_i)$ because Z and X are d-separated. Also $P(E_{Yi\backslash X}^+)$ is a constant

$$P(E_X^- \mid X) =$$

$$\prod_i \sum_{y_i} P(E_{Yi}^- \mid y_i) \sum_{z_i} \beta_i P(z_i \mid E_{Yi\backslash X}^+) P(y_i \mid X, z_i)$$

The computation of $P(E_X^- \mid X)$

$$P(E_X^- \mid X) =$$

$$\prod_i \sum_{y_i} P(E_{Yi}^- \mid y_i) \sum_{z_i} \beta_i P(z_i \mid E_{Yi\backslash X}^+) P(y_i \mid X, z_i)$$

- The parents of Y_i (the Z_{ij}) are independent of each other.
- We also combine the β_i into one single β

The computation of $P(E_X^- \mid X)$

$$P(E_X^- \mid X) = \beta \prod_i \sum_{y_i} P(E_{Yi}^- \mid y_i) \sum_{z_i} P(y_i \mid X, z_i) \prod_j P(z_{ij} \mid E_{Z_{ij} \setminus Y_i})$$

- $P(E_{Yi}^- \mid y_i)$ is a recursive instance of $P(E_X^- \mid X)$
- $P(y_i \mid X, z_i)$ is a cond prob table entry for Y_i
- $P(z_{ij} \mid E_{Zij \setminus Yi})$ is a recursive sub-instance of the $P(X \mid E)$ calculation

Inference in multiply connected belief networks

- Clustering methods
  - Transform the net into a probabilistically equivalent (but topologically different) poly-tree by merging offending nodes
- Conditioning methods
  - Instantiate variables to definite values, and then evaluate a poly-tree for each possible instantiation

Inference in multiply connected belief networks

- Stochastic simulation methods
  - Use the network to generate a large number of concrete models of the domain that are consistent with the network distribution.
  - They give an approximation of the exact evaluation.

Default reasoning

- Some conclusions are made by default unless a counter-evidence is obtained
  - Non-monotonic reasoning
- Points to ponder
  - Whats the semantic status of default rules?
  - What happens when the evidence matches the premises of two default rules with conflicting conclusions?
  - If a belief is retracted later, how can a system keep track of which conclusions need to be retracted as a consequence?

Issues in Rule-based methods for Uncertain Reasoning

- Locality
  - In logical reasoning systems, if we have $A \Rightarrow B$, then we can conclude B given evidence A, *without worrying about any other rules*. In probabilistic systems, we need to consider *all* available evidence.

Issues in Rule-based methods for Uncertain Reasoning

- Detachment
  - Once a logical proof is found for proposition B, we can use it regardless of how it was derived (*it can be detached from its justification*). In probabilistic reasoning, the source of the evidence is important for subsequent reasoning.

Issues in Rule-based methods for Uncertain Reasoning

- Truth functionality
  - In logic, the truth of complex sentences can be computed from the truth of the components. Probability combination does not work this way, except under strong independence assumptions.

A famous example of a truth functional system for uncertain reasoning is the *certainty factors model*, developed for the Mycin medical diagnostic program

Dempster-Shafer Theory

- Designed to deal with the distinction between *uncertainty* and *ignorance*.
- We use a belief function *Bel(X)* – probability that the evidence supports the proposition
- When we do not have any evidence about X, we assign Bel(X) = 0 as well as Bel(¬X) = 0

Dempster-Shafer Theory

For example, if we do not know whether a coin is fair, then:

$Bel(\text{ Heads }) = Bel(\neg\text{Heads }) = 0$

If we are given that the coin is fair with 90% certainty, then:

$Bel(\text{ Heads }) = 0.9 \times 0.5 = 0.45$

$Bel(\neg\text{Heads }) = 0.9 \times 0.5 = 0.45$

Note that we still have a gap of 0.1 *that is not accounted for by the evidence*

Fuzzy Logic

- Fuzzy set theory is a means of specifying how well an object satisfies a vague description
  - Truth is a value between 0 and 1
  - Uncertainty stems from lack of evidence, but given the dimensions of a man concluding whether he is fat has no uncertainty involved

Fuzzy Logic

- The rules for evaluating the fuzzy truth, T, of a complex sentence are

$$T(A \wedge B) = \min(\ T(A), T(B)\ )$$

$$T(A \vee B) = \max(\ T(A), T(B)\ )$$

$$T(\neg A) = 1 - T(A)$$