Complexity and Correctness of LTL Model Checking

Lecture #17 of Model Checking

Joost-Pieter Katoen

Lehrstuhl 2: Software Modeling & Verification

E-mail: `katoen@cs.rwth-aachen.de`

December 17, 2008

Overview Lecture #17

⇒ Repetition: from LTL to GNBA

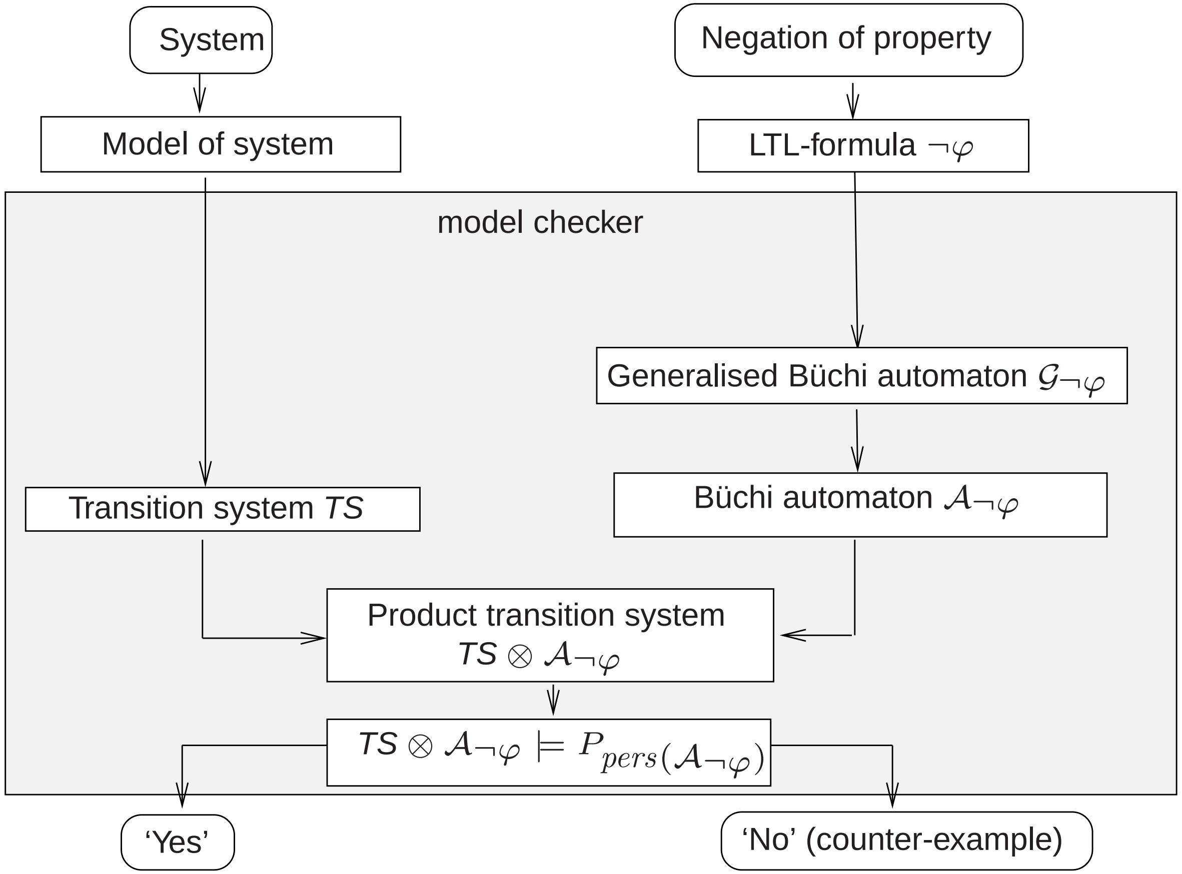
- Correctness proof
- Complexity results
  - LTL model checking is coNP-hard and PSPACE-complete
  - Satisfiability and validity are PSPACE-hard
- Summary of LTL model checking

Reduction to persistence checking

$$
\begin{aligned}
TS \models \varphi \quad & \text{if and only if} \quad \mathit{Traces}(TS) \subseteq \mathit{Words}(\varphi) \\
& \text{if and only if} \quad \mathit{Traces}(TS) \cap \left((2^{AP})^{\omega} \setminus \mathit{Words}(\varphi)\right) = \varnothing \\
& \text{if and only if} \quad \mathit{Traces}(TS) \cap \underbrace{\mathit{Words}(\neg\varphi)}_{\mathcal{L}_{\omega}(\mathcal{A}_{\neg\varphi})} = \varnothing \\
& \text{if and only if} \quad TS \otimes \mathcal{A}_{\neg\varphi} \models \Diamond\Box\neg F
\end{aligned}
$$

LTL model checking is thus reduced to persistence checking!

Overview of LTL model checking



From LTL to GNBA

GNBA $\mathcal{G}_\varphi$ over 2^{AP} for LTL-formula φ with $\mathcal{L}_\omega(\mathcal{G}_\varphi) = \mathit{Words}(\varphi)$:

- Assume φ only contains the operators $\wedge$, $\neg$, $\bigcirc$ and U
  - $\vee$, $\rightarrow$, $\Diamond$, $\Box$, W, and so on, are expressed in terms of these basic operators
- States are *elementary sets* of sub-formulas in φ
  - for $\sigma = A_0 A_1 A_2 \ldots \in \mathit{Words}(\varphi)$, expand $A_i \subseteq AP$ with sub-formulas of φ
  - ... to obtain the infinite word $\bar{\sigma} = B_0 B_1 B_2 \ldots$ such that

$$\psi \in B_i \qquad \text{if and only if} \qquad \sigma^i = A_i A_{i+1} A_{i+2} \ldots \models \psi$$

  - $\bar{\sigma}$ is intended to be a run in GNBA $\mathcal{G}_\varphi$ for σ
- Transitions are derived from semantics $\bigcirc$ and expansion law for U
- Accept sets guarantee that: $\bar{\sigma}$ is an accepting run for σ iff $\sigma \models \varphi$

Elementary sets of formulae

$B \subseteq \mathit{closure}(\varphi)$ is *elementary* if:

1. B is *logically consistent* if for all $\varphi_1 \wedge \varphi_2, \psi \in \mathit{closure}(\varphi)$:
   - $\varphi_1 \wedge \varphi_2 \in B \;\Leftrightarrow\; \varphi_1 \in B$ and $\varphi_2 \in B$
   - $\psi \in B \;\Rightarrow\; \neg\psi \notin B$
   - true $\in \mathit{closure}(\varphi) \;\Rightarrow\;$ true $\in B$

2. B is *locally consistent* if for all $\varphi_1 \,\mathsf{U}\, \varphi_2 \in \mathit{closure}(\varphi)$:
   - $\varphi_2 \in B \;\Rightarrow\; \varphi_1 \,\mathsf{U}\, \varphi_2 \in B$
   - $\varphi_1 \,\mathsf{U}\, \varphi_2 \in B$ and $\varphi_2 \notin B \;\Rightarrow\; \varphi_1 \in B$

3. B is *maximal*, i.e., for all $\psi \in \mathit{closure}(\varphi)$:
   - $\psi \notin B \;\Rightarrow\; \neg\psi \in B$

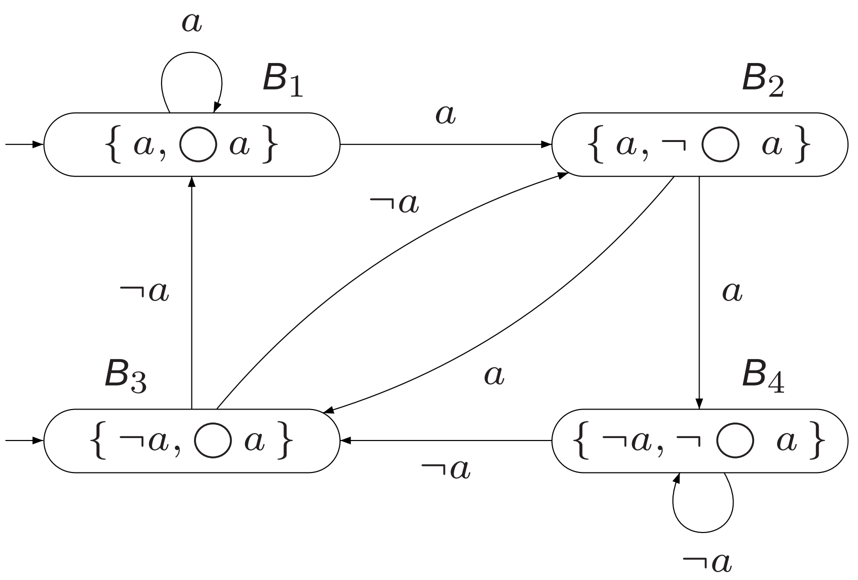
The GNBA of LTL-formula φ

For LTL-formula φ, let $\mathcal{G}_\varphi = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$ where

- Q = all elementary sets $B \subseteq \mathit{closure}(\varphi)$, $Q_0 = \{ B \in Q \mid \varphi \in B \}$
- $\mathcal{F} = \{ \{ B \in Q \mid \varphi_1 \mathsf{U} \varphi_2 \notin B \text{ or } \varphi_2 \in B \} \mid \varphi_1 \mathsf{U} \varphi_2 \in \mathit{closure}(\varphi) \}$
- The transition relation $\delta : Q \times 2^{AP} \to 2^Q$ is given by:
  - If $A \neq B \cap AP$ then $\delta(B, A) = \varnothing$
  - $\delta(B, B \cap AP)$ is the set of all elementary sets of formulas B' satisfying:
    (i) For every $\bigcirc \psi \in \mathit{closure}(\varphi)$: $\bigcirc \psi \in B \Leftrightarrow \psi \in B'$, and
    (ii) For every $\varphi_1 \mathsf{U} \varphi_2 \in \mathit{closure}(\varphi)$:

$$\varphi_1 \mathsf{U} \varphi_2 \in B \Leftrightarrow \Big(\varphi_2 \in B \vee (\varphi_1 \in B \wedge \varphi_1 \mathsf{U} \varphi_2 \in B') \Big)$$

GNBA for LTL-formula $\bigcirc a$



$Q_0 = \{ B_1, B_3 \}$ since $\bigcirc a \in B_1$ and $\bigcirc a \in B_3$

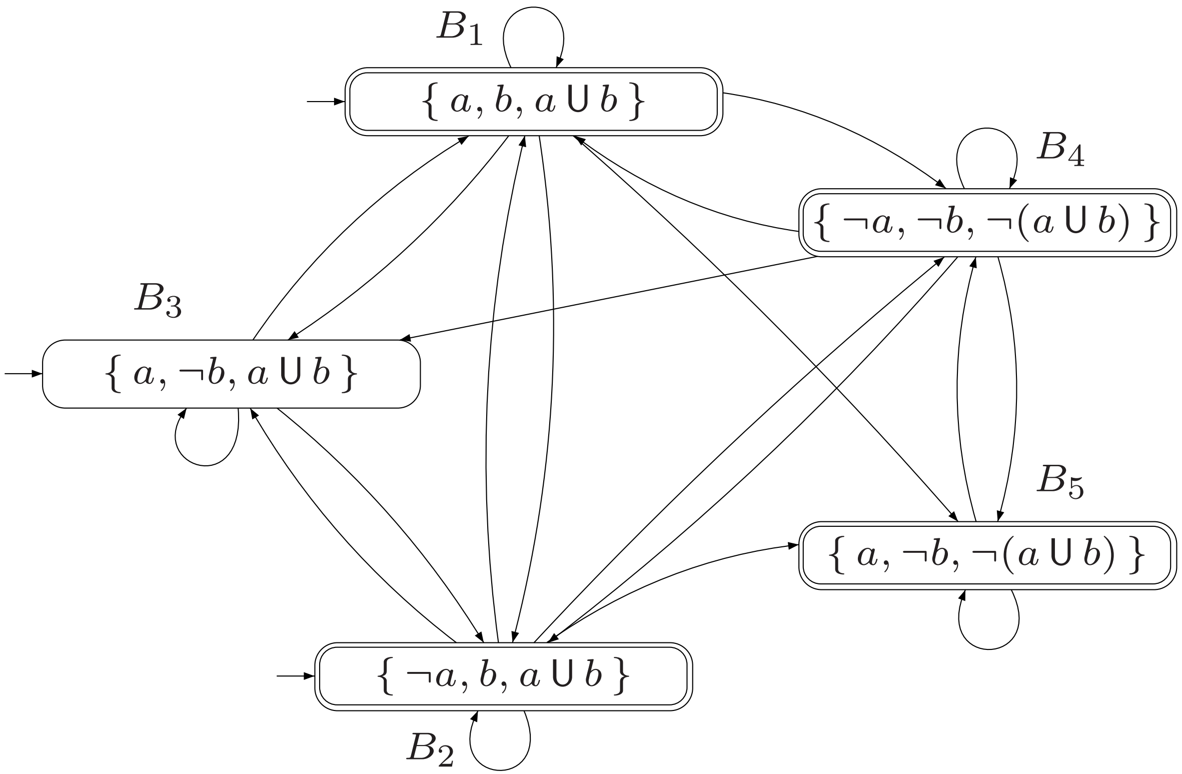
$\delta(B_2, \{ a \}) = \{ B_3, B_4 \}$ as $B_2 \cap \{ a \} = \{ a \}$, $\neg \bigcirc a = \bigcirc \neg a \in B_2$, and $\neg a \in B_3, B_4$

$\delta(B_1, \{ a \}) = \{ B_1, B_2 \}$ as $B_1 \cap \{ a \} = \{ a \}$, $\bigcirc a \in B_1$ and $a \in B_1, B_2$

$\delta(B_4, \{ a \}) = \varnothing$ since $B_4 \cap \{ a \} = \varnothing \neq \{ a \}$

The set $\mathcal{F}$ is empty, since $\varphi = \bigcirc a$ does not contain an until-operator

GNBA for LTL-formula $a \,\mathsf{U}\, b$



justification: on the black board

Overview Lecture #17

- Repetition: from LTL to GNBA

$\Rightarrow$ Correctness proof

- Complexity results
  - LTL model checking is coNP-hard and PSPACE-complete
  - Satisfiability and validity are PSPACE-hard

- Summary of LTL model checking

Correctness theorem

$$\boxed{\mathit{Words}(\varphi) = \mathcal{L}_\omega(\mathcal{G}_\varphi)}$$

Proof: on the black board

NBA are more expressive than LTL

Corollary: every LTL-formula expresses an ω-regular property

But: there exist ω-regular properties that cannot be expressed in LTL

Example: there is no LTL formula φ with *Words*$(\varphi) = P$ for the LT-property:

$$P \;=\; \left\{ A_0 A_1 A_2 \ldots \in \left(2^{\{\,a\,\}}\right)^{\omega} \mid a \in A_{2i} \text{ for } i \geqslant 0 \right\}$$

But there exists an NBA $\mathcal{A}$ with $\mathcal{L}_\omega(\mathcal{A}) = P$

$\Rightarrow$ *there are ω-regular properties that cannot be expressed in LTL!*

Overview Lecture #16

- Repetition: from LTL to GNBA
- Correctness proof

⇒ Complexity results
  - LTL model checking is coNP-hard and PSPACE-complete
  - Satisfiability and validity are PSPACE-hard

- Summary of LTL model checking

Complexity for LTL to NBA

For any LTL-formula φ (over *AP*) there exists an NBA $\mathcal{A}_\varphi$

with $\mathit{Words}(\varphi) = \mathcal{L}_\omega(\mathcal{A}_\varphi)$ and

which can be constructed in time and space in $2^{\mathcal{O}(|\varphi| \cdot \log |\varphi|)}$

Justification complexity: next slide

Time and space complexity in $2^{\mathcal{O}(|\varphi| \cdot \log |\varphi|)}$

- States GNBA $\mathcal{G}_\varphi$ are elementary sets of formulae in *closure*(φ)
  - sets B can be represented by bit vectors with single bit per subformula ψ of φ
- The number of states in $\mathcal{G}_\varphi$ is bounded by $2^{|\text{subf}(\varphi)|}$
  - where subf(φ) denotes the set of all subformulae of φ
  - $|\text{subf}(\varphi)| \leqslant 2 \cdot |\varphi|$; so, the number of states in $\mathcal{G}_\varphi$ is bounded by $2^{\mathcal{O}(|\varphi|)}$
- The number of accepting sets of $\mathcal{G}_\varphi$ is bounded above by $\mathcal{O}(|\varphi|)$
- The number of states in NBA $\mathcal{A}_\varphi$ is thus bounded by $2^{\mathcal{O}(|\varphi)|} \cdot \mathcal{O}(|\varphi|)$
- $2^{\mathcal{O}(|\varphi|)} \cdot \mathcal{O}(|\varphi|) \;=\; 2^{\mathcal{O}(|\varphi| \log |\varphi|)}$ qed

Lower bound

There exists a family of LTL formulas φ_n with $|\varphi_n| = \mathcal{O}(poly(n))$

such that every NBA $\mathcal{A}_{\varphi_n}$ for φ_n has at least 2^n states

Proof (1)

Let *AP* be non-empty, that is, $|2^{AP}| \geqslant 2$ and:

$$\mathcal{L}_n = \left\{ A_1 \ldots A_n A_1 \ldots A_n \sigma \mid A_i \subseteq AP \wedge \sigma \in \left(2^{AP}\right)^{\omega} \right\}, \qquad \text{for } n \geqslant 0$$

It follows $\mathcal{L}_n = \mathit{Words}(\varphi_n)$ where $\varphi_n = \bigwedge_{a \in AP} \bigwedge_{0 \leqslant i < n} (\bigcirc^i a \longleftrightarrow \bigcirc^{n+i} a)$

φ_n is an LTL formula of polynomial length: $|\varphi_n| \in \mathcal{O}\Big(|AP| \cdot n\Big)$

However, any NBA $\mathcal{A}$ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_n$ has at least 2^n states

Proof (2)

Claim: any NBA $\mathcal{A}$ for $\bigwedge_{a \in AP} \bigwedge_{0 \leqslant i < n} (\bigcirc^{i} a \longleftrightarrow \bigcirc^{n+i} a)$ has at least 2^n states

Words of the form $A_1 \ldots A_n \, A_1 \ldots A_n \, \varnothing \, \varnothing \, \varnothing \ldots$ are accepted by $\mathcal{A}$

$\mathcal{A}$ thus has for every word $A_1 \ldots A_n$ of length n, a state $q(A_1 \ldots A_n)$, say, which can be reached from an initial state by consuming $A_1 \ldots A_n$

From $q(A_1 \ldots A_n)$, it is possible to visit an accept state infinitely often by accepting the suffix $A_1 \ldots A_n \, \varnothing \, \varnothing \, \varnothing \ldots$

If $A_1 \ldots A_n \neq A'_1 \ldots A'_n$ then

$$A_1 \ldots A_n \, A'_1 \ldots A'_n \, \varnothing \, \varnothing \, \varnothing \ldots \notin \mathcal{L}_n = \mathcal{L}_\omega(\mathcal{A})$$

Therefore, the states $q(A_1 \ldots A_n)$ are all pairwise different

Given $|2^{AP}|$ possible sequences $A_1 \ldots A_n$, NBA $\mathcal{A}$ has $\geqslant \left(\left| 2^{AP} \right| \right)^n \geqslant 2^n$ states

Complexity for LTL model checking

The time and space complexity of LTL model checking is in $\mathcal{O}\left(|TS| \cdot 2^{|\varphi|}\right)$

On-the-fly LTL model checking

- Idea: find a counter-example *during* the generation of $\mathit{Reach}(\mathit{TS})$ and $\mathcal{A}_{\neg\varphi}$
  - exploit the fact that $\mathit{Reach}(\mathit{TS})$ and $\mathcal{A}_{\neg\varphi}$ can be generated in parallel

$\Rightarrow$ Generate $\mathit{Reach}(\mathit{TS} \otimes \mathcal{A}_{\neg\varphi})$ “on demand”

  - consider a new vertex only if no accepting cycle has been found yet
  - only consider the successors of a state in $\mathcal{A}_{\neg\varphi}$ that match current state in TS

$\Rightarrow$ Possible to find an accepting cycle without generating $\mathcal{A}_{\neg\varphi}$ entirely

- This on-the-fly scheme is adopted in e.g. the model checker SPIN

The LTL model-checking problem is co-NP-hard

The Hamiltonian path problem is polynomially reducible to
the complement of the LTL model-checking problem

In fact, the LTL model-checking problem is PSPACE-complete [Sistla & Clarke 1985]

LTL satisfiability and validity checking

- Satisfiability problem: $\mathit{Words}(\varphi) \neq \varnothing$ for LTL-formula φ?
  - does there exist a transition system for which φ holds?
- Solution: construct an NBA $\mathcal{A}_\varphi$ and check for emptiness
  - nested depth-first search for checking persistence properties
- Validity problem: is $\varphi \equiv \text{true}$, i.e., $\mathit{Words}(\varphi) = \left(2^{AP}\right)^\omega$?
  - does φ hold for every transition system?
- Solution: as for satisfiability, as φ is valid iff $\neg\varphi$ is satisfiable

run time is exponential; a more efficient algorithm most probably does not exist!

LTL satisfiability and validity checking

$\boxed{\text{The satisfiability and validity problem for LTL are PSPACE-complete}}$

Black board: show the fact that these problems are PSPACE-hard

Overview Lecture #16

- Repetition: from LTL to GNBA

- Correctness proof

- Complexity results
  - LTL model checking is coNP-hard and PSPACE-complete
  - Satisfiability and validity are PSPACE-hard

$\Rightarrow$ Summary of LTL model checking

Summary of LTL model checking (1)

- LTL is a logic for formalizing path-based properties
- Expansion law allows for rewriting until into local conditions and next
- LTL-formula φ can be transformed algorithmically into NBA $\mathcal{A}_\varphi$
  - this may cause an exponential blow up
  - algorithm: first construct a GNBA for φ; then transform it into an equivalent NBA
- LTL-formulae describe ω-regular LT-properties
  - but do not have the same expressivity as ω-regular languages

Summary of LTL model checking (2)

- $TS \models \varphi$ can be solved by a nested depth-first search in $TS \otimes \mathcal{A}_{\neg\varphi}$
  - time complexity of the LTL model-checking algorithm is linear in TS and exponential in $|\varphi|$

- Fairness assumptions can be described by LTL-formulae

  the model-checking problem for LTL with fairness is reducible
  to the standard LTL model-checking problem

- The LTL-model checking problem is PSPACE-complete

- Satisfiability and validity of LTL amounts to NBA emptiness-check

- The satisfiability and validity problem for LTL are PSPACE-complete