Linear-Time Properties

Lecture #5b of Model Checking

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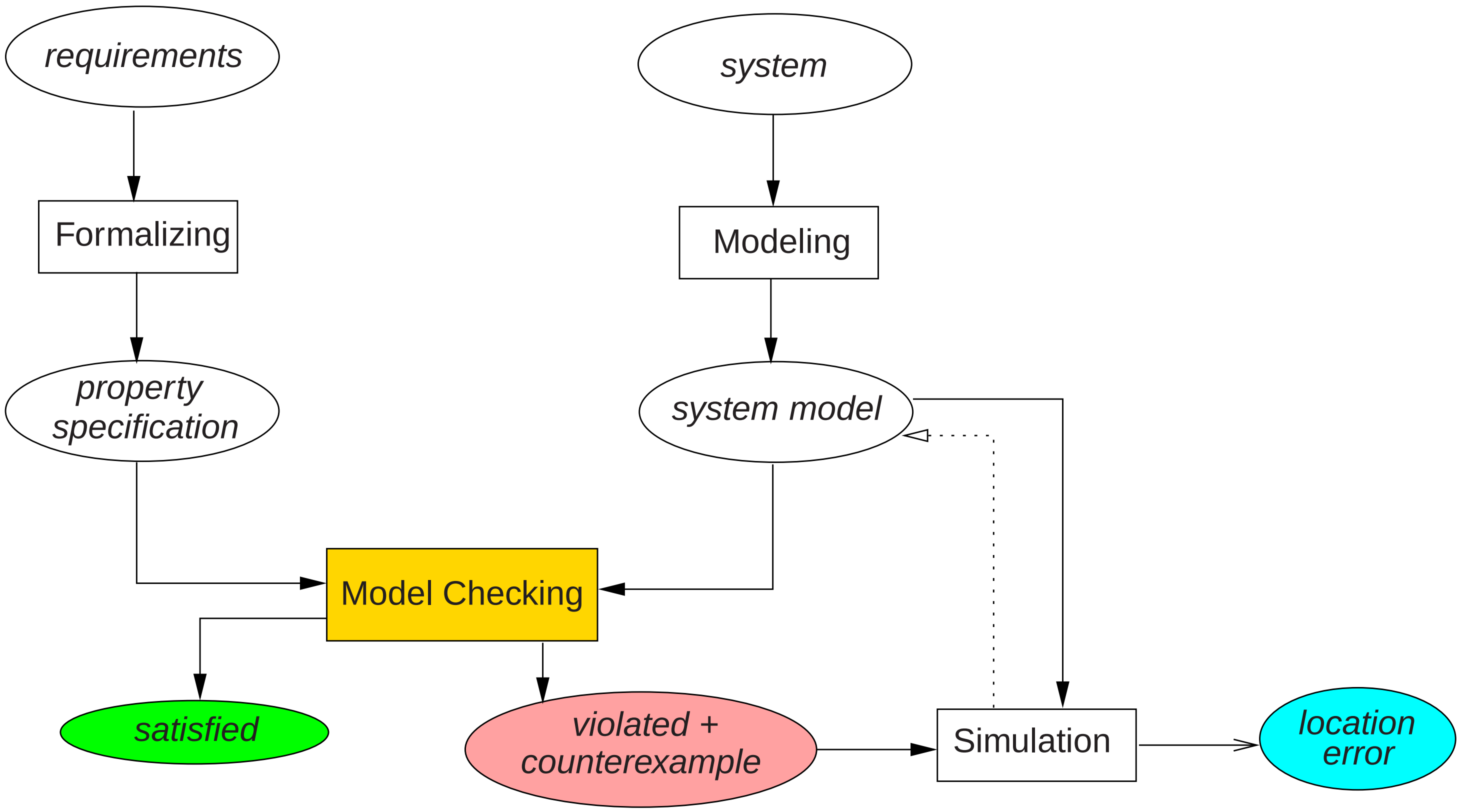
Overview Lecture #5

- Paths and traces
- Linear-time (LT) properties
- Trace equivalence and LT properties
- Invariants

Recall model checking



we now consider: what are properties?

Recall executions

- A *finite execution fragment* ϱ of *TS* is an alternating sequence of states and actions ending with a state:

$$\varrho \;=\; s_0\,\alpha_1\,s_1\,\alpha_2\,\ldots\,\alpha_n\,s_n \text{ such that } s_i \xrightarrow{\alpha_{i+1}} s_{i+1} \text{ for all } 0 \leqslant i < n.$$

- An *infinite execution fragment* ρ of *TS* is an infinite, alternating sequence of states and actions:

$$\rho \;=\; s_0\,\alpha_1\,s_1\,\alpha_2\,s_2\,\alpha_3\ldots \text{ such that } s_i \xrightarrow{\alpha_{i+1}} s_{i+1} \text{ for all } 0 \leqslant i.$$

- An *execution* of *TS* is an initial, maximal execution fragment
  - a *maximal* execution fragment is either finite ending in a terminal state, or infinite
  - an execution fragment is *initial* if $s_0 \in I$

State graph

- The *state graph* of *TS*, notation $G(\mathit{TS})$, is the digraph (V, E)

  with vertices $V = S$ and edges $E = \{(s, s') \in S \times S \mid s' \in \mathit{Post}(s)\}$

  $\Rightarrow$ omit all state and transition labels in *TS* and ignore being initial

- $\mathit{Post}^*(s)$ is the set of states reachable $G(\mathit{TS})$ from s

$$\mathit{Post}^*(C) = \bigcup_{s \in C} \mathit{Post}^*(s) \quad \text{for } C \subseteq S$$

- The notations $\mathit{Pre}^*(s)$ and $\mathit{Pre}^*(C)$ have analogous meaning

- The set of reachable states: $\mathit{Reach}(\mathit{TS}) = \mathit{Post}^*(I)$

Path fragments

- A path fragment is an execution fragment without actions
- A *finite path fragment* $\widehat{\pi}$ of *TS* is a state sequence:

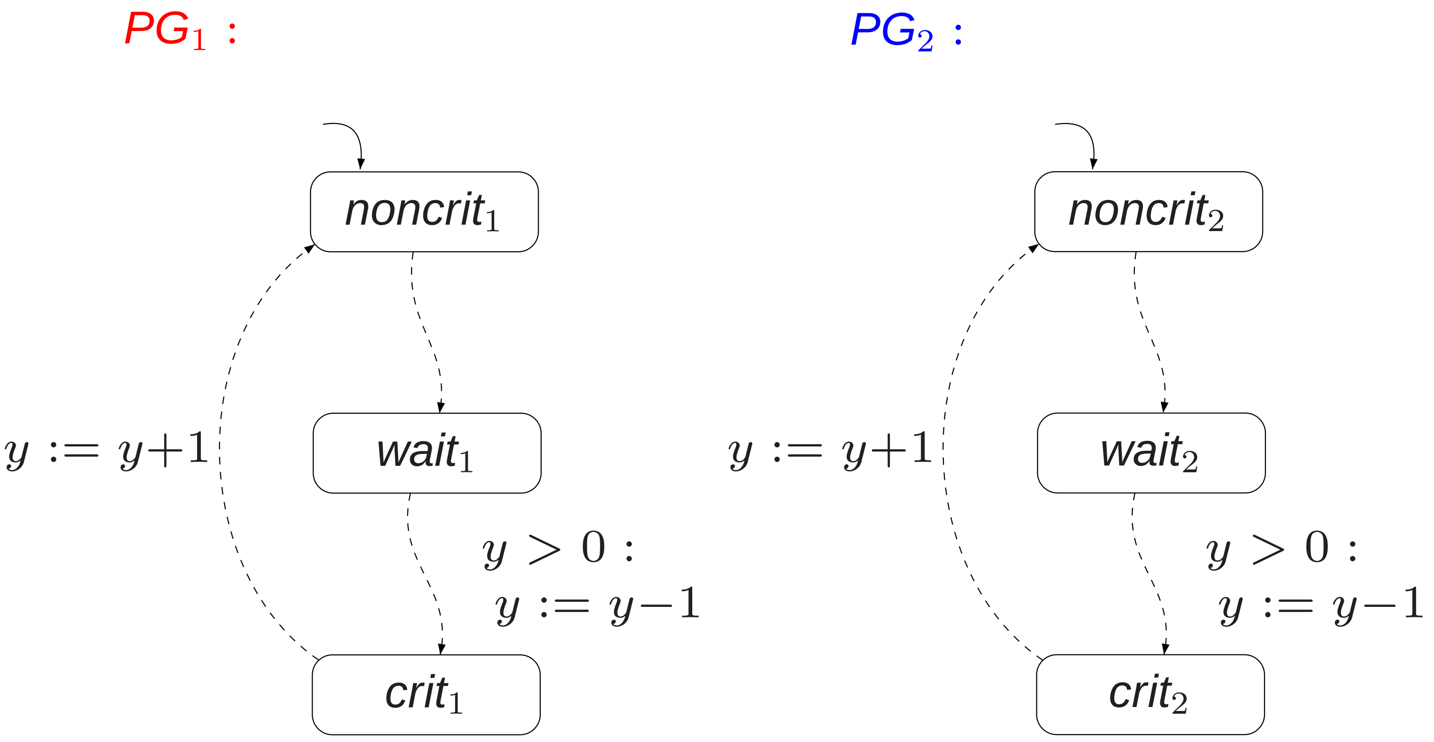
$$\widehat{\pi} = s_0 \, s_1 \ldots s_n \quad \text{such that} \quad s_{i+1} \in \mathit{Post}(s_i) \text{ for all } 0 \leqslant i < n \text{ where } n \geqslant 0$$

- An *infinite path fragment* π of *TS* is an infinite state sequence:

$$\pi = s_0 \, s_1 \, s_2 \ldots \quad \text{such that } s_{i+1} \in \mathit{Post}(s_i) \text{ for all } i \geqslant 0$$

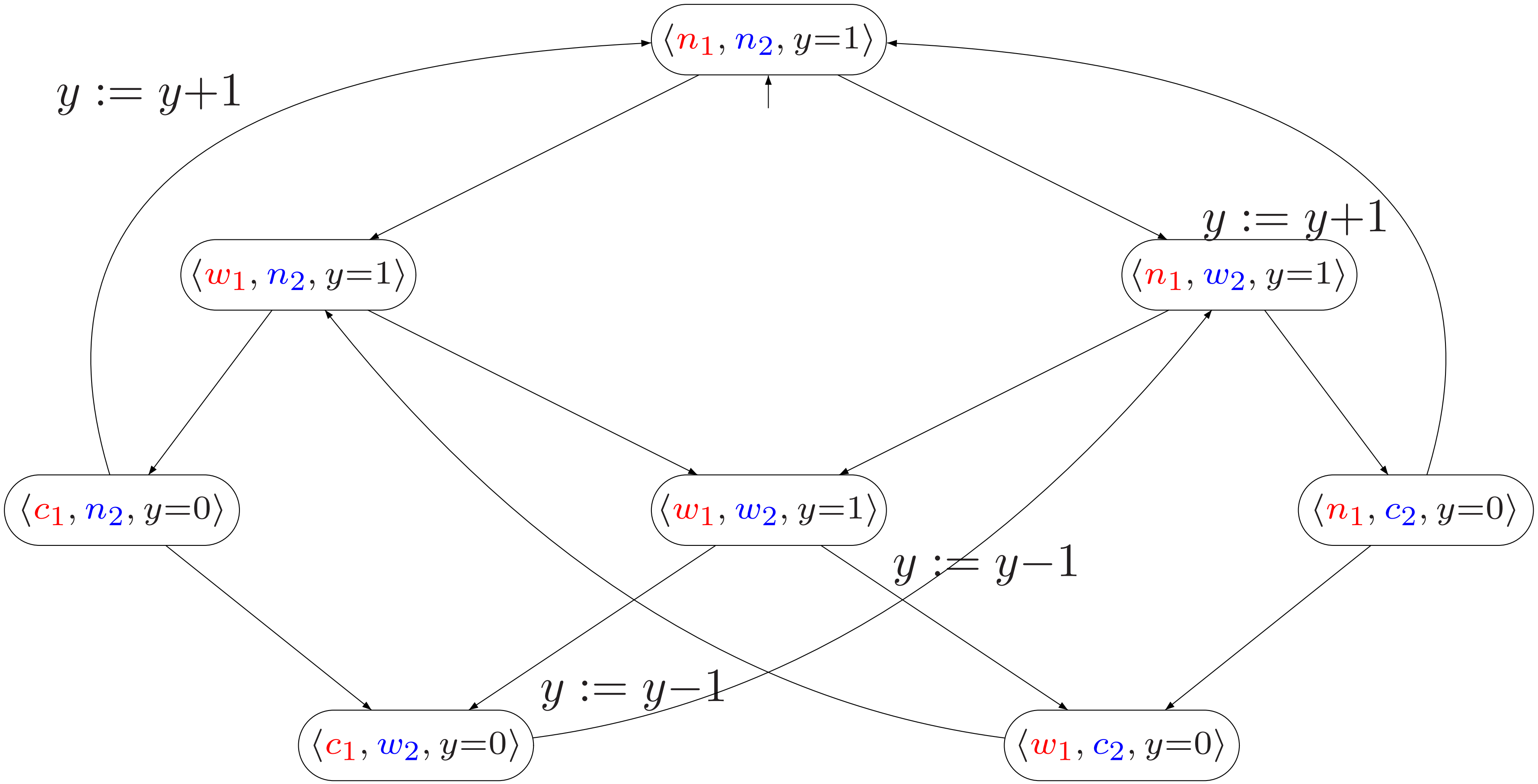
- A *path* of *TS* is an initial, maximal path fragment
  - a *maximal* path fragment is either finite ending in a terminal state, or infinite
  - a path fragment is *initial* if $s_0 \in I$
  - *Paths*(s) is the set of maximal path fragments π with *first*$(\pi) = s$

Semaphore-based mutual exclusion



$y=0$ means “lock is currently possessed”; $y=1$ means “lock is free”

Transition system $TS(PG_1 \,|||\, PG_2)$



Example paths

Traces

- Actions are mainly used to model the (possibility of) interaction
  - synchronous or asynchronous communication
- Here, focus on the states that are visited during executions
  - the states themselves are not "observable", but just their atomic propositions
- Consider sequences of the form $L(s_0)\, L(s_1)\, L(s_2) \ldots$
  - just register the (set of) atomic propositions that are valid along the execution
  - instead of execution $s_0 \xrightarrow{\alpha_0} s_1 \xrightarrow{\alpha_1} s_2 \ldots$

  $\Rightarrow$ this is called a *trace*
- For a transition system without terminal states:
  - traces are infinite words over the alphabet 2^{AP}, i.e., they are in $\left(2^{AP}\right)^{\omega}$

Traces

- Let transition system *TS* $= (S, Act, \rightarrow, I, AP, L)$ without terminal states
  - all maximal paths (and excutions) are infinite
- The *trace* of path fragment $\pi = s_0\, s_1 \ldots$ is $trace(\pi) = L(s_0)\, L(s_1) \ldots$
  - the trace of $\widehat{\pi} = s_0\, s_1 \ldots s_n$ is $trace(\widehat{\pi}) = L(s_0)\, L(s_1) \ldots L(s_n)$
- The set of traces of a set Π of paths: $trace(\Pi) = \{\, trace(\pi) \mid \pi \in \Pi \,\}$
- $Traces(s) = trace(Paths(s))$ $\qquad$ $Traces(TS) = \bigcup_{s \in I} Traces(s)$
- $Traces_{fin}(s) = trace(Paths_{fin}(s))$ $\qquad$ $Traces_{fin}(TS) = \bigcup_{s \in I} Traces_{fin}(s)$

Example traces

Let $AP = \{ \mathit{crit}_1, \mathit{crit}_2 \}$

Example path:

$$\begin{aligned} \pi \;=\; & \langle n_1, n_2, y=1\rangle \rightarrow \langle w_1, n_2, y=1\rangle \rightarrow \langle c_1, n_2, y=0\rangle \rightarrow \\ & \langle n_1, n_2, y=1\rangle \rightarrow \langle n_1, w_2, y=1\rangle \rightarrow \langle n_1, c_2, y=0\rangle \rightarrow \ldots \end{aligned}$$

The trace of this path is the infinite word:

$$\mathit{trace}(\pi) \;=\; \varnothing\,\varnothing\,\{ \mathit{crit}_1 \}\,\varnothing\,\varnothing\,\{ \mathit{crit}_2 \}\,\varnothing\,\varnothing\,\{ \mathit{crit}_1 \}\,\varnothing\,\varnothing\,\{ \mathit{crit}_2 \}\ldots$$

The trace of the finite path fragment:

$$\begin{aligned} \widehat{\pi} \;=\; & \langle n_1, n_2, y=1\rangle \rightarrow \langle w_1, n_2, y=1\rangle \rightarrow \langle w_1, w_2, y=1\rangle \rightarrow \\ & \langle w_1, c_2, y=0\rangle \rightarrow \langle w_1, n_2, y=1\rangle \rightarrow \langle c_1, n_2, y=0\rangle \end{aligned}$$

is:

$$\mathit{trace}(\widehat{\pi}) \;=\; \varnothing\,\varnothing\,\varnothing\,\{ \mathit{crit}_2 \}\,\varnothing\,\{ \mathit{crit}_1 \}$$

Linear-time properties

- Linear-time properties specify the traces that a TS must exhibit
  - LT-property specifies the admissible behaviour of system under consideration

later, a logic will be introduced for specifying LT properties

- A *linear-time property* (LT property) over *AP* is a subset of $\left(2^{AP}\right)^{\omega}$
  - finite words are not needed, as it is assumed that there are *no terminal states*

- *TS* (over *AP*) *satisfies* LT property P (over *AP*):

$$TS \models P \quad \text{if and only if} \quad \mathit{Traces}(TS) \subseteq P$$

  - *TS* satisfies the LT property P if all its “observable” behaviors are admissible
  - state $s \in S$ satisfies P, notation $s \models P$, whenever $\mathit{Traces}(s) \subseteq P$

How to specify mutual exclusion?

"Always at most one process is in its critical section"

- Let $AP = \{ \mathit{crit}_1, \mathit{crit}_2 \}$
  - other atomic propositions are not of any relevance for this property
- Formalization as LT property

$$P_{mutex} \; = \; \text{set of infinite words } A_0\, A_1\, A_2 \ldots \text{ with } \{ \mathit{crit}_1, \mathit{crit}_2 \} \not\subseteq A_i \text{ for all } 0 \leqslant i$$

- Contained in P_{mutex} are e.g., the infinite words:
  - $(\{ \mathit{crit}_1 \} \{ \mathit{crit}_2 \})^\omega$ and $\{ \mathit{crit}_1 \} \{ \mathit{crit}_1 \} \{ \mathit{crit}_1 \} \ldots$ and $\varnothing \varnothing \varnothing \ldots$
  - but not $\{ \mathit{crit}_1 \} \varnothing \{ \mathit{crit}_1, \mathit{crit}_2 \} \ldots$ or $\varnothing \{ \mathit{crit}_1 \}, \varnothing \varnothing \{ \mathit{crit}_1, \mathit{crit}_2 \} \varnothing \ldots$

Does the semaphore-based algorithm satisfy P_{mutex}?

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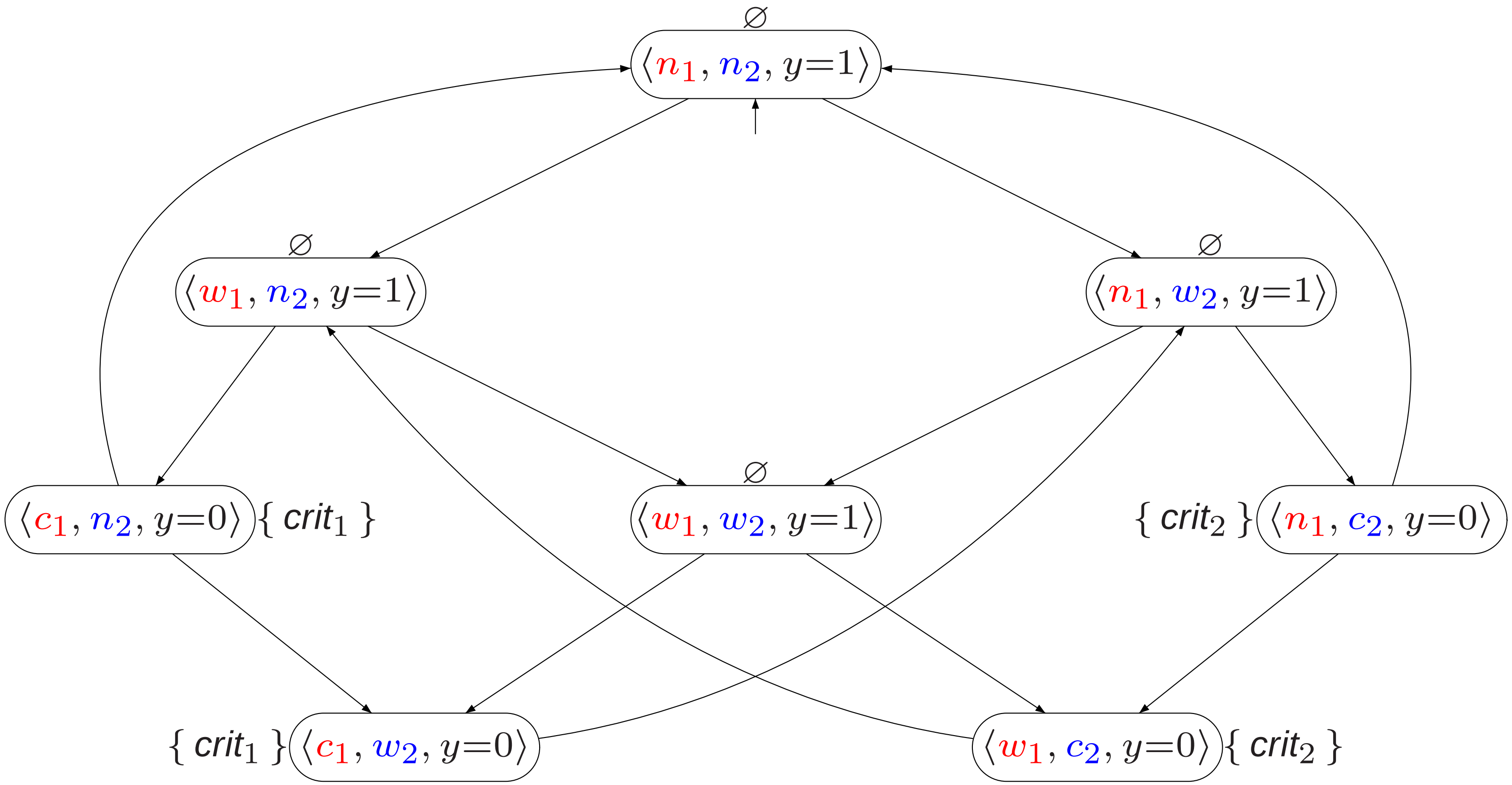

Yes as there is no reachable state labeled with $\{ \textit{crit}_1, \textit{crit}_2 \}$

How to specify starvation freedom?

"A process that wants to enter the critical section is eventually able to do so'"

- Let $AP = \{ \mathit{wait}_1, \mathit{crit}_1, \mathit{wait}_2, \mathit{crit}_2 \}$
- Formalization as LT-property

$P_{nostarve} =$ set of infinite words $A_0\, A_1\, A_2 \ldots$ such that:

$$\left(\overset{\infty}{\exists}\ j.\ \mathit{wait}_i \in A_j \right) \;\Rightarrow\; \left(\overset{\infty}{\exists}\ j.\ \mathit{crit}_i \in A_j \right) \quad \text{for each } i \in \{ 1, 2 \}$$

there exist infinitely many: $\left(\overset{\infty}{\exists}\, j.\, \mathit{wait}_i \in A_j \right) \equiv (\forall k \geqslant 0.\, \exists j > k.\, \mathit{wait}_i \in A_j)$

Does the semaphore-based algorithm satisfy $P_{nostarve}$?

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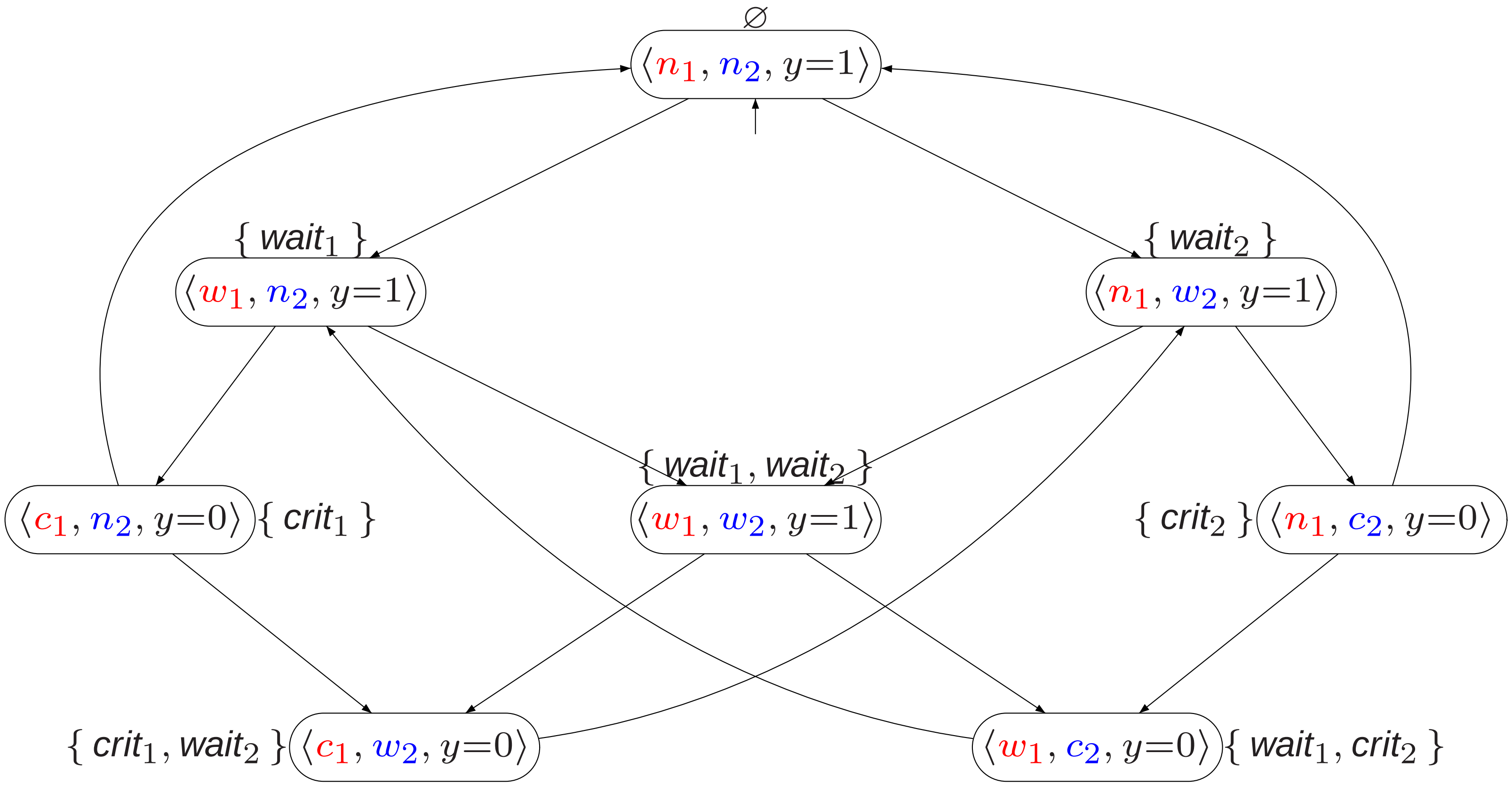
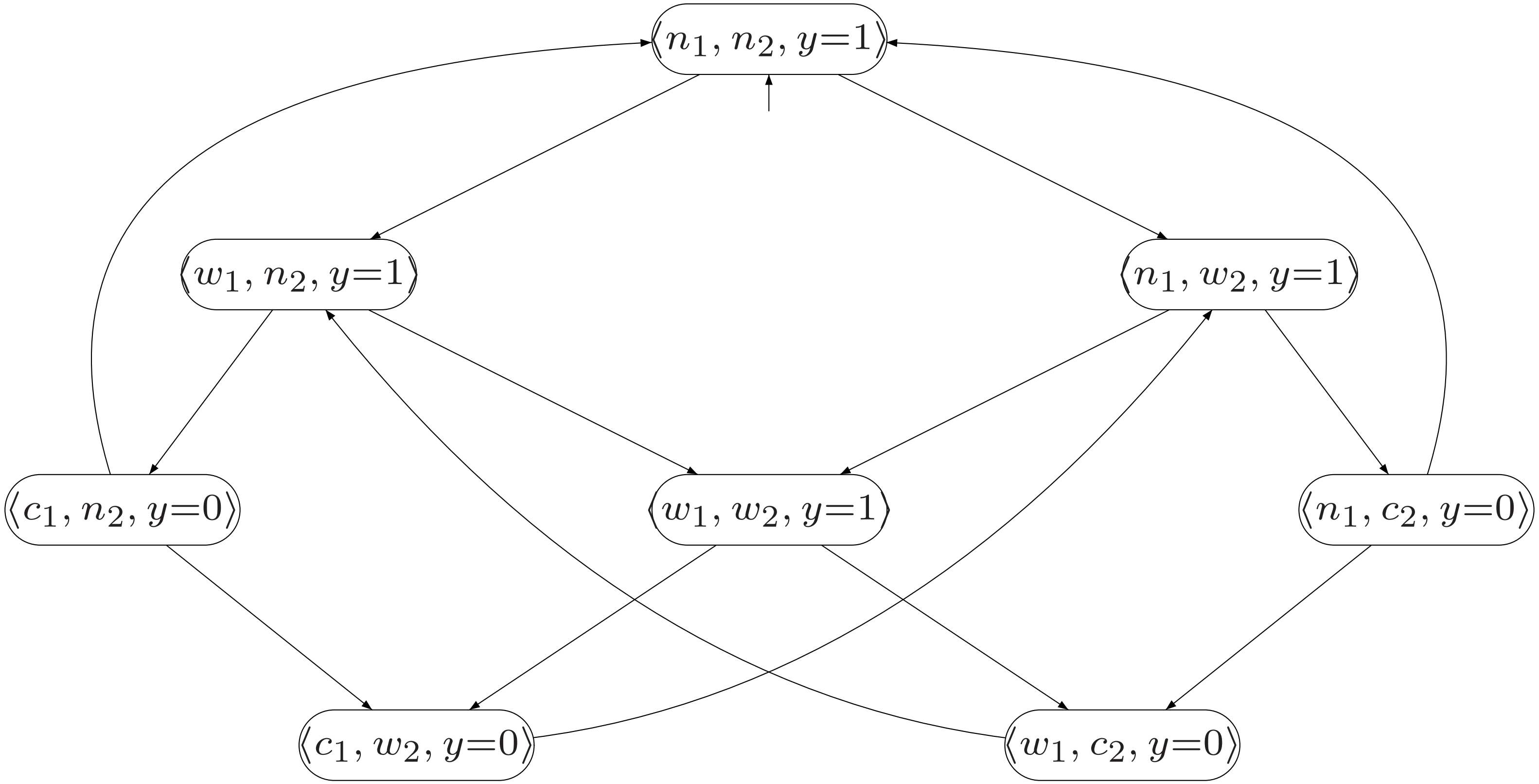


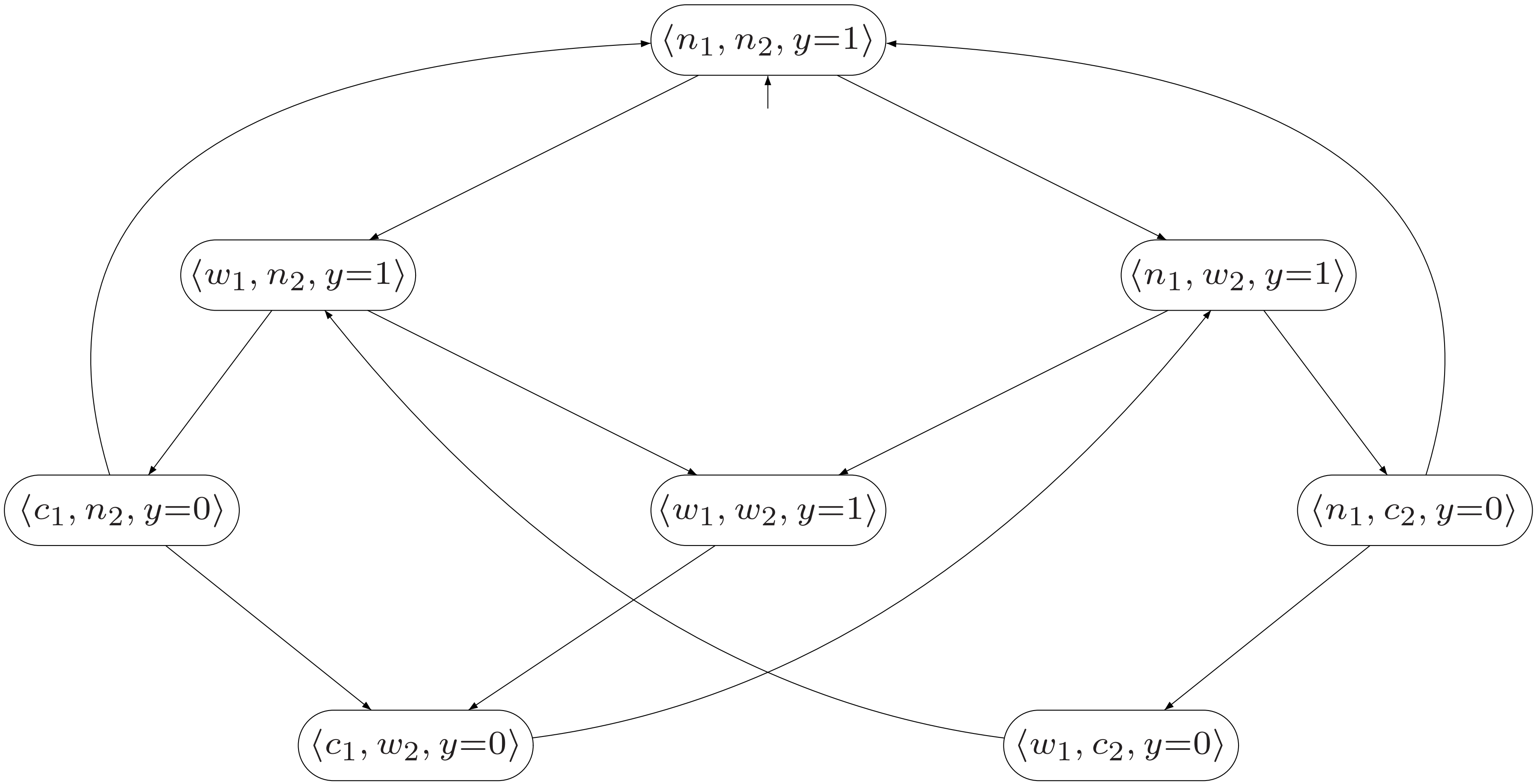
No. Trace $\varnothing\ (\{\,wait_2\,\}\ \{\,wait_1, wait_2\,\}\ \{\,crit_1, wait_2\,\}\,)^\omega \in Traces(TS)$, but $\notin P_{nostarve}$

Mutual exclusion algorithm revisited



this algorithm satisfies P_{mutex}

Refining mutual exclusion algorithm



this variant algorithm with an omitted edge also satisfies P_{mutex}

Trace equivalence and LT properties

For *TS* and *TS*$'$ be transition systems (over *AP*) without terminal states:

$$\mathit{Traces}(\mathit{TS}) \subseteq \mathit{Traces}(\mathit{TS}')$$

if and only if

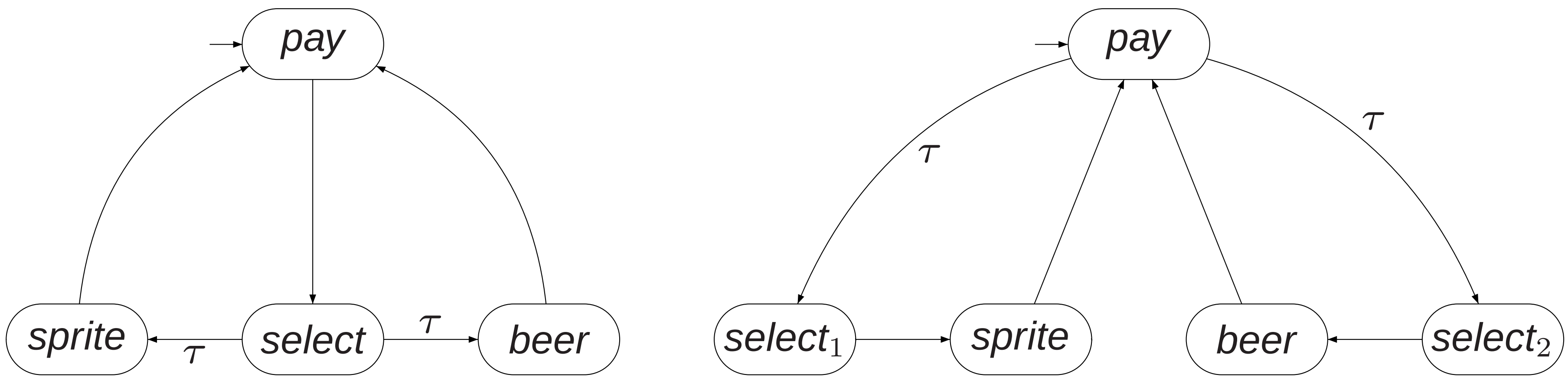
for any LT property P: $\mathit{TS}' \models P$ implies $\mathit{TS} \models P$

$$\mathit{Traces}(\mathit{TS}) = \mathit{Traces}(\mathit{TS}')$$

if and only if

TS and *TS*$'$ satisfy the same LT properties

Two beverage vending machines



$$AP = \{ pay, sprite, beer \}$$

there is no LT-property that can distinguish between these machines

Invariants

- Safety properties $\approx$ "nothing bad should happen" [Lamport 1977]
- Typical safety property: mutual exclusion property
  - the bad thing (having > 1 process in the critical section) never occurs
- Another typical safety property is deadlock freedom

$\Rightarrow$ These properties are in fact invariants

- An invariant is an LT property
  - that is given by a condition Φ for the states
  - and requires that Φ holds for all reachable states
  - e.g., for mutex property $\Phi \equiv \neg crit_1 \ \vee \ \neg crit_2$

Invariants

- An LT property P_{inv} over *AP* is an *invariant* if there is a propositional logic formula Φ over *AP* such that:

$$P_{inv} = \left\{ A_0 A_1 A_2 \ldots \in \left(2^{AP}\right)^{\omega} \mid \forall j \geqslant 0.\ A_j \models \Phi \right\}$$

  - Φ is called an *invariant condition* of P_{inv}

- Note that

$$\begin{array}{lll} TS \models P_{inv} & \text{iff} & trace(\pi) \in P_{inv} \text{ for all paths } \pi \text{ in } TS \\ & \text{iff} & L(s) \models \Phi \text{ for all states } s \text{ that belong to a path of } TS \\ & \text{iff} & L(s) \models \Phi \text{ for all states } s \in Reach(TS) \end{array}$$

- Φ has to be fulfilled by all initial states and
  - satisfaction of Φ is invariant under all transitions in the reachable fragment of *TS*

Checking an invariant

- Checking an invariant for the propositional formula Φ

  $=$ check the validity of Φ in every reachable state
  $\Rightarrow$ use a slight modification of standard graph traversal algorithms (DFS and BFS)
  - provided the given transition system *TS* is *finite*

- Perform a forward depth-first search

  - at least one state s is found with $s \not\models \Phi \Rightarrow$ the invariance of Φ is violated

- Alternative: backward search

  - starts with all states where Φ does not hold
  - calculates (by a DFS or BFS) the set $\bigcup_{s \in S, s \not\models \Phi} \mathit{Pre}^*(s)$

A naive invariant checking algorithm

Input: finite transition system *TS* and propositional formula Φ
Output: true if *TS* satisfies the invariant "always Φ", otherwise false

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set of state $R := \varnothing$; (* the set of visited states *)
stack of state $U := \varepsilon$; (* the empty stack *)
bool $b :=$ true; (* all states in R satisfy Φ *)
for all $s \in I$ **do**
  if $s \notin R$ **then**
    visit(s) (* perform a dfs for each unvisited initial state *)
  fi
od
return b

A naive invariant checking algorithm

procedure visit (state s)
  push(s, U); (* push s on the stack *)
  $R := R \cup \{ s \}$; (* mark s as reachable *)
  repeat
    $s' :=$ *top*(U);
    if *Post*$(s') \subseteq R$ **then**
      pop(U);
      $b := b \ \wedge \ (s' \models \Phi)$; (* check validity of Φ in s' *)
    else
      let $s'' \in$ *Post*$(s') \setminus R$
      push(s'', U);
      $R := R \cup \{ s'' \}$; (* state s'' is a new reachable state *)
    fi
  until $(U = \varepsilon)$
endproc

error indication is state refuting Φ

initial path fragment $s_0 \, s_1 \, s_2 \ldots s_n$ *with* $s_i \models \Phi$ *(*$i \neq n$*) and* $s_n \not\models \Phi$ *is more useful*

Invariant checking by DFS

Input: finite transition system *TS* and propositional formula Φ
Output: "yes" if *TS* $\models$ "always Φ", otherwise "no" plus a counterexample

```
set of states R := ∅;                         (* the set of reachable states *)
stack of states U := ε;                       (* the empty stack *)
bool b := true;                               (* all states in R satisfy Φ *)
while (I \ R ≠ ∅ ∧ b) do
  let s ∈ I \ R;                              (* choose an arbitrary initial state not in R *)
  visit(s);                                   (* perform a DFS for each unvisited initial state *)
od
if b then
  return("yes")                               (* TS ⊨ "always Φ" *)
else
  return("no", reverse(U))                    (* counterexample arises from the stack content *)
fi
```

Invariant checking by DFS

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    else
      let $s'' \in$ *Post*$(s') \setminus R$
      push(s'', U);
      $R := R \cup \{ s'' \}$; (* state s'' is a new reachable state *)
    fi
  until $((U = \varepsilon) \ \vee \ \neg b)$
endproc

Time complexity

- Under the assumption that
  - $s' \in \mathit{Post}(s)$ can be encountered in time $\Theta(|\mathit{Post}(s)|)$

  $\Rightarrow$ this holds for a representation of $\mathit{Post}(s)$ by adjacency lists

- The time complexity for invariant checking is $\mathcal{O}(\,N * (1 + |\Phi|) + M\,)$
  - where N denotes the number of reachable states, and
  - $M = \sum_{s \in S} |\mathit{Post}(s)|$ the number of transitions in the reachable fragment of *TS*

- The adjacency lists are typically given *implicitly*
  - e.g., by a syntactic description of the concurrent processes as program graphs
  - $\mathit{Post}(s)$ is obtained by the rules for the transition relation