Transition Systems

Lecture #2 of Model Checking

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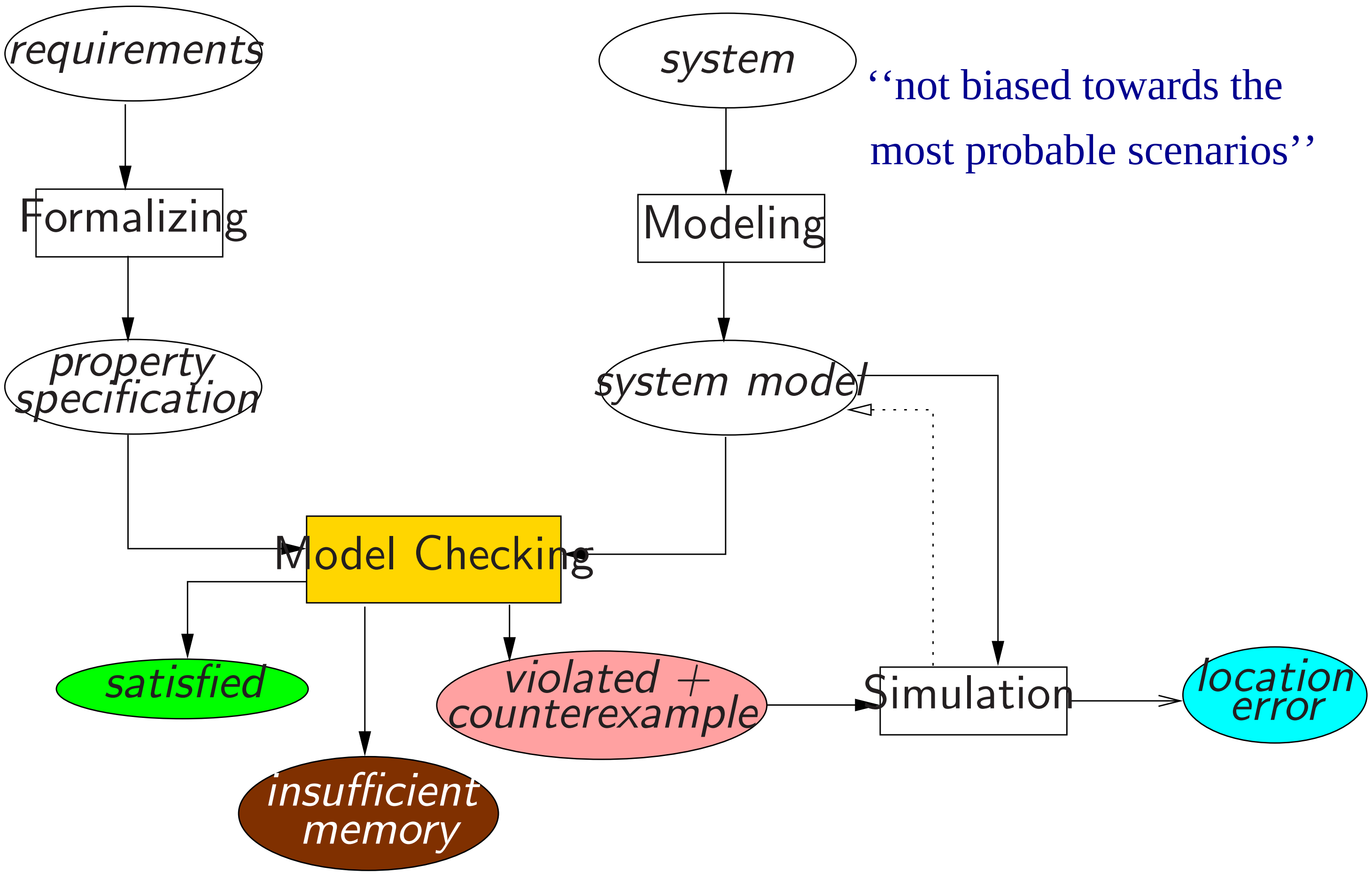
Overview Lecture #2

⇒ *Transition systems*

- Executions
- Modeling data-dependent systems

- *Parallelism and communication*
  - Interleaving
  - Shared variables

Recall model checking



''not biased towards the most probable scenarios''

Transition systems

- model to describe the behaviour of systems
- digraphs where nodes represent *states*, and edges model *transitions*
- state:
  - the current colour of a traffic light
  - the current values of all program variables + the program counter
  - the current value of the registers together with the values of the input bits
- transition: ("state change")
  - a switch from one colour to another
  - the execution of a program statement
  - the change of the registers and output bits for a new input

Transition system

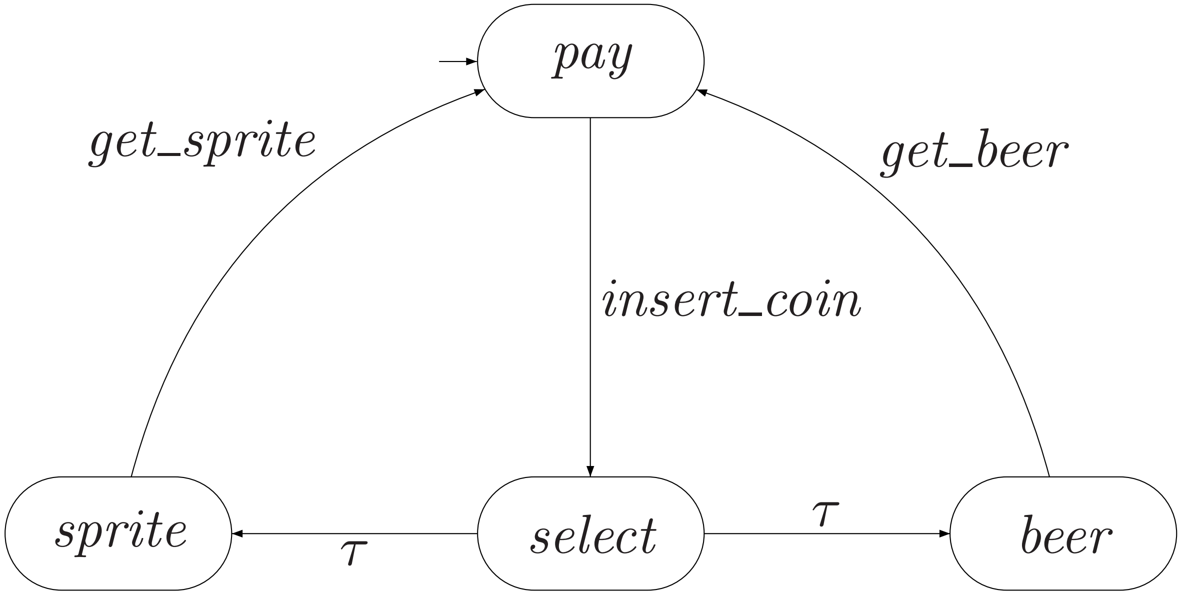
A *transition system* TS is a tuple $(S, Act, \to, I, AP, L)$ where

- S is a set of states
- Act is a set of actions
- $\longrightarrow \subseteq S \times Act \times S$ is a transition relation
- $I \subseteq S$ is a set of initial states
- AP is a set of atomic propositions
- $L : S \to 2^{AP}$ is a labeling function

S and Act are either finite or countably infinite

Notation: $s \xrightarrow{\alpha} s'$ instead of $(s, \alpha, s') \in \longrightarrow$

A beverage vending machine



states? actions?, transitions?, initial states?

Atomic propositions?

Direct successors and predecessors

$$Post(s,\alpha) = \left\{ s' \in S \mid s \xrightarrow{\alpha} s' \right\}, \qquad Post(s) = \bigcup_{\alpha \in Act} Post(s,\alpha)$$

$$Pre(s,\alpha) = \left\{ s' \in S \mid s' \xrightarrow{\alpha} s \right\}, \qquad Pre(s) = \bigcup_{\alpha \in Act} Pre(s,\alpha).$$

$$Post(C,\alpha) = \bigcup_{s \in C} Post(s,\alpha), \quad Post(C) = \bigcup_{s \in C} Post(s) \text{ for } C \subseteq S.$$

$$Pre(C,\alpha) = \bigcup_{s \in C} Pre(s,\alpha), \quad Pre(C) = \bigcup_{s \in C} Pre(s) \text{ for } C \subseteq S.$$

State s is called *terminal* if and only if $Post(s) = \varnothing$

Action- and *AP*-determinism

Transition system $\mathit{TS} = (S, \mathit{Act}, \rightarrow, I, \mathit{AP}, L)$ is *action-deterministic* iff:

$$|\, I \,| \;\leqslant\; 1 \quad \text{and} \quad |\, \mathit{Post}(s, \alpha) \,| \;\leqslant\; 1 \qquad \text{for all } s, \alpha$$

Transition system $\mathit{TS} = (S, \mathit{Act}, \rightarrow, I, \mathit{AP}, L)$ is *AP-deterministic* iff:

$$|\, I \,| \;\leqslant\; 1 \;\text{ and }\; |\, \underbrace{\mathit{Post}(s) \,\cap\, \{\, s' \in S \mid L(s') = A \,\}}_{\text{equally labeled successors of } s} | \;\leqslant\; 1 \qquad \text{for all } s, A \in 2^{\mathit{AP}}$$

The role of nondeterminism

Here: nondeterminism is a feature!

- to model concurrency by interleaving
  - no assumption about the relative speed of processes
- to model implementation freedom
  - only describes what a system should do, not how
- to model under-specified systems, or abstractions of real systems
  - use incomplete information

in automata theory, nondeterminism may be exponentially more succinct
but that's not the issue here!

Executions

- A *finite execution fragment* ϱ of *TS* is an alternating sequence of states and actions ending with a state:

$$\varrho \;=\; s_0\,\alpha_1\,s_1\,\alpha_2\,\ldots\alpha_n\,s_n \text{ such that } s_i \xrightarrow{\alpha_{i+1}} s_{i+1} \text{ for all } 0 \leqslant i < n.$$

- An *infinite execution fragment* ρ of *TS* is an infinite, alternating sequence of states and actions:

$$\rho \;=\; s_0\,\alpha_1\,s_1\,\alpha_2\,s_2\,\alpha_3\ldots \text{ such that } s_i \xrightarrow{\alpha_{i+1}} s_{i+1} \text{ for all } 0 \leqslant i.$$

- An *execution* of *TS* is an initial, maximal execution fragment
  - a *maximal* execution fragment is either finite ending in a terminal state, or infinite
  - an execution fragment is *initial* if $s_0 \in I$

Example executions

$$\rho_1 = pay \xrightarrow{coin} select \xrightarrow{\tau} sprite \xrightarrow{sget} pay \xrightarrow{coin} select \xrightarrow{\tau} sprite \xrightarrow{sget} \ldots$$

$$\rho_2 = select \xrightarrow{\tau} sprite \xrightarrow{sget} pay \xrightarrow{coin} select \xrightarrow{\tau} beer \xrightarrow{bget} \ldots$$

$$\varrho = pay \xrightarrow{coin} select \xrightarrow{\tau} sprite \xrightarrow{sget} pay \xrightarrow{coin} select \xrightarrow{\tau} sprite$$

Execution fragments ρ_1 and ϱ are initial, but ρ_2 is not

ϱ is not maximal as it does not end in a terminal state

Assuming that ρ_1 and ρ_2 are infinite, they are maximal

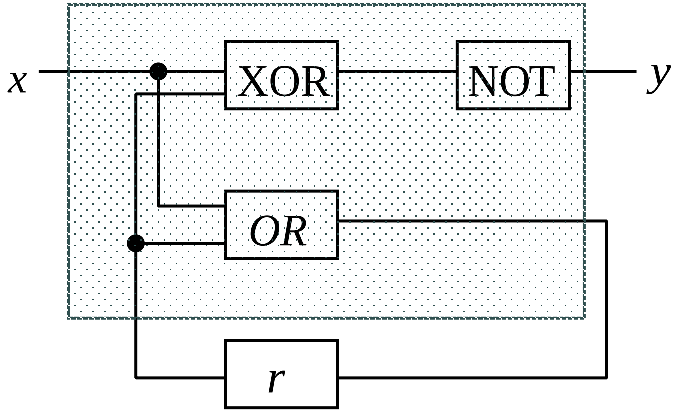
Reachable states

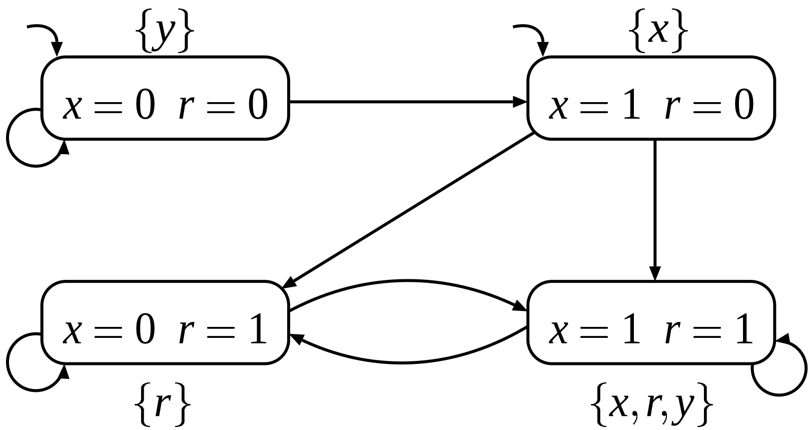
State $s \in S$ is called *reachable* in *TS* if there exists an initial, finite execution fragment

$$s_0 \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_n} s_n = s\ .$$

Reach(*TS*) denotes the set of all reachable states in *TS*.

Modeling sequential circuits





Transition system representation of a simple hardware circuit

Input variable x, output variable y, and register r

Output function $\neg(x \oplus r)$ and register evaluation function $x \vee r$

Atomic propositions

Consider two possible state-labelings:

- Let $AP = \{ x, y, r \}$
  - $L(\langle x = 0, r = 1 \rangle) = \{ r \}$ and $L(\langle x = 1, r = 1 \rangle) = \{ x, r, y \}$
  - $L(\langle x = 0, r = 0 \rangle) = \{ y \}$ and $L(\langle x = 1, r = 0 \rangle) = \{ x \}$
  - property e.g., "once the register is one, it remains one"

- Let $AP' = \{ x, y \}$ – the register evaluations are now "invisible"
  - $L(\langle x = 0, r = 1 \rangle) = \varnothing$ and $L(\langle x = 1, r = 1 \rangle) = \{ x, y \}$
  - $L(\langle x = 0, r = 0 \rangle) = \{ y \}$ and $L(\langle x = 1, r = 0 \rangle) = \{ x \}$
  - property e.g., "the output bit y is set infinitely often"

Beverage vending machine revisited

"Abstract" transitions:

$$start \xrightarrow{true:coin} select \quad \text{and} \quad start \xrightarrow{true:refill} start$$

$$select \xrightarrow{nsprite>0:sget} start \quad \text{and} \quad select \xrightarrow{nbeer>0:bget} start$$

$$select \xrightarrow{nsprite=0 \wedge nbeer=0:ret\_coin} start$$

Action	Effect on variables
$coin$	
ret_coin	
$sget$	$nsprite := nsprite - 1$
$bget$	$nbeer := nbeer - 1$
$refill$	$nsprite := max;\ nbeer := max$

Program graph representation

Some preliminaries

- typed variables with a valuation that assigns values to variables
  - e.g., $\eta(x) = 17$ and $\eta(y) = -2$
- the set of Boolean conditions over *Var*
  - propositional logic formulas whose propositions are of the form "$\overline{x} \in \overline{D}$"
  - $(-3 < x \leqslant 5) \ \wedge \ (y = green) \ \wedge \ (x \leqslant 2 \cdot x')$
- effect of the actions is formalized by means of a mapping:

$$\mathit{Effect} : \mathit{Act} \times \mathit{Eval}(\mathit{Var}) \rightarrow \mathit{Eval}(\mathit{Var})$$

  - e.g., $\alpha \equiv x := y+5$ and evaluation $\eta(x) = 17$ and $\eta(y) = -2$
  - $\mathit{Effect}(\alpha, \eta)(x) \ = \ \eta(y) + 5 = 3,$ and $\mathit{Effect}(\alpha, \eta)(y) \ = \eta(y) = -2$

Program graphs

A *program graph* *PG* over set *Var* of typed variables is a tuple

$$(\mathit{Loc}, \mathit{Act}, \mathit{Effect}, \longrightarrow, \mathit{Loc}_0, g_0) \quad \text{where}$$

- Loc is a set of *locations* with initial locations $\mathit{Loc}_0 \subseteq \mathit{Loc}$
- Act is a set of actions
- $\mathit{Effect} : \mathit{Act} \times \mathit{Eval}(\mathit{Var}) \to \mathit{Eval}(\mathit{Var})$ is the *effect* function
- $\longrightarrow \subseteq \mathit{Loc} \times (\underbrace{\mathit{Cond}(\mathit{Var})}_{\text{Boolean conditions over } \mathit{Var}} \times \mathit{Act}) \times \mathit{Loc}$, transition relation
- $g_0 \in \mathit{Cond}(\mathit{Var})$ is the initial *condition*.

Notation: $\ell \xrightarrow{g:\alpha} \ell'$ denotes $(\ell, g, \alpha, \ell') \in \longrightarrow$

Beverage vending machine

- $\mathit{Loc} = \{\, start, select \,\}$ with $\mathit{Loc}_0 = \{\, start \,\}$
- $\mathit{Act} = \{\, bget, sget, coin, ret\_coin, refill \,\}$
- $\mathit{Var} = \{\, nsprite, \;\; nbeer \,\}$ with domain $\{\, 0, 1, \ldots, max \,\}$
- $$\begin{array}{lcl} \mathit{Effect}(coin, \eta) & = & \eta \\ \mathit{Effect}(ret\_coin, \eta) & = & \eta \\ \mathit{Effect}(sget, \eta) & = & \eta[nsprite := nsprite{-}1] \\ \mathit{Effect}(bget, \eta) & = & \eta[nbeer := nbeer{-}1] \\ \mathit{Effect}(refill, \eta) & = & [nsprite := max, nbeer := max] \end{array}$$
- $g_0 = (nsprite = max \;\wedge\; nbeer = max)$

From program graphs to transition systems

- Basic strategy: *unfolding*
  - state = location (current control) ℓ + data valuation η
  - initial state = initial location satisfying the initial condition g_0

- Propositions and labeling
  - propositions: "at ℓ" and "$x \in D$" for $D \subseteq \mathit{dom}(x)$
  - $\langle \ell, \eta \rangle$ is labeled with "at ℓ" and all conditions that hold in η

- $\ell \xrightarrow{g:\alpha} \ell'$ and g holds in η then $\langle \ell, \eta \rangle \xrightarrow{\alpha} \langle \ell', \mathit{Effect}(\alpha, \eta) \rangle$

Structured operational semantics

- The notation $\dfrac{\text{premise}}{\text{conclusion}}$ means:

- If the proposition above the “solid line” (i.e., the premise) holds, then the proposition under the fraction bar (i.e., the conclusion) holds

- Such “if . . ., then . . .” propositions are also called *inference rules*

- If the premise is a tautology, it may be omitted (as well as the “solid line”)

- In the latter case, the rule is also called an *axiom*

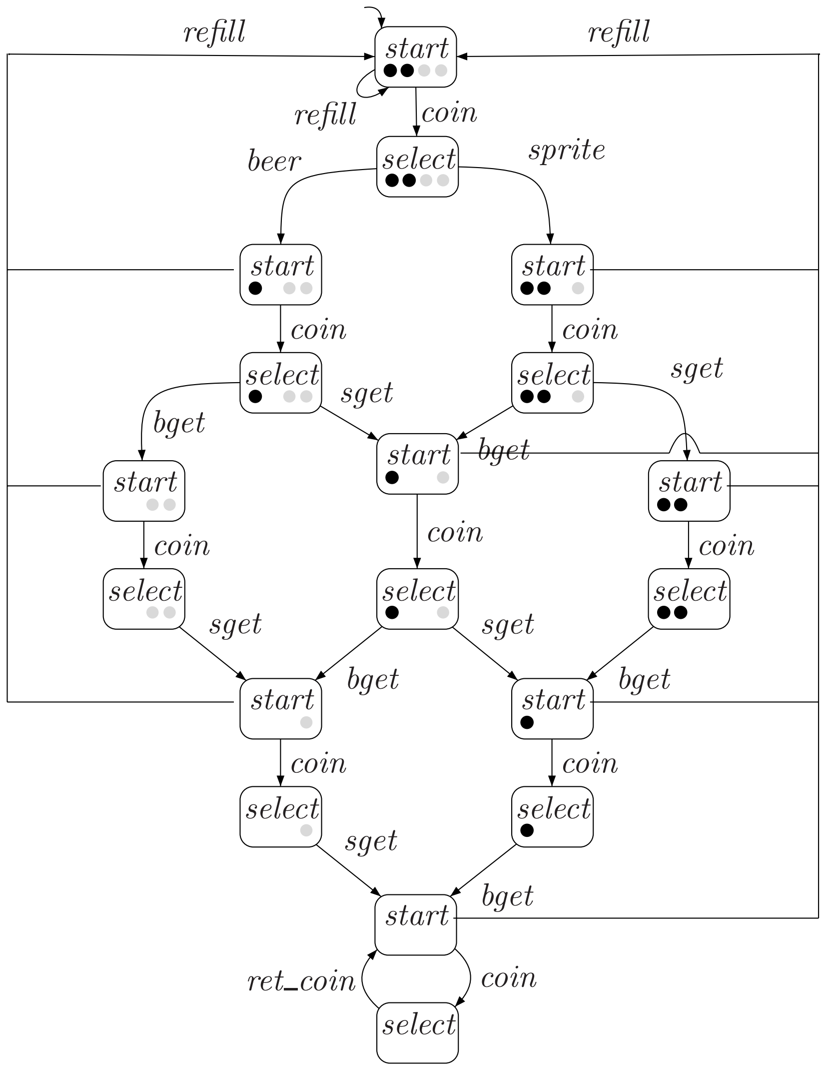
Transition systems for program graphs

The transition system $\mathit{TS}(\mathit{PG})$ of program graph

$$\mathit{PG} = (\mathit{Loc}, \mathit{Act}, \mathit{Effect}, \longrightarrow, \mathit{Loc}_0, g_0)$$

over set Var of variables is the tuple $(S, \mathit{Act}, \longrightarrow, I, \mathit{AP}, L)$ where

- $S = \mathit{Loc} \times \mathit{Eval}(\mathit{Var})$
- $\longrightarrow \subseteq S \times Act \times S$ is defined by the rule: $\dfrac{\ell \xrightarrow{g:\alpha} \ell' \quad \wedge \quad \eta \models g}{\langle \ell, \eta \rangle \xrightarrow{\alpha} \langle \ell', \mathit{Effect}(\alpha, \eta) \rangle}$
- $I = \{ \langle \ell, \eta \rangle \mid \ell \in \mathit{Loc}_0, \eta \models g_0 \}$
- $\mathit{AP} = \mathit{Loc} \cup \mathit{Cond}(\mathit{Var})$ and $L(\langle \ell, \eta \rangle) = \{\ell\} \cup \{ g \in \mathit{Cond}(\mathit{Var}) \mid \eta \models g \}$.


refill
refill
start
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sget
bget
start
ret_coin
coin
select

Transition systems $\neq$ finite automata

As opposed to finite automata, in a transition system:

- there are *no* accept states
- set of states and actions may be countably infinite
- may have infinite branching
- actions may be subject to synchronization (cf. next lecture)
- nondeterminism has a different role

Transition systems are appropriate for reactive system behaviour