Regular Languages

Foundations of Computing Science

Pallab Dasgupta
Professor,
Dept. of Computer Sc & Engg

Deterministic Finite Automaton (DFA)

A *deterministic finite automaton (DFA)* is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- Q is a finite set called the set of *states*,
- Σ is a finite set called the *alphabet*,
- $\delta: Q \times \Sigma \rightarrow Q$ is the *transition function*,
- $q_0 \in Q$ is the *start state*, and
- $F \subseteq Q$ is the *set of accept states (final states)*

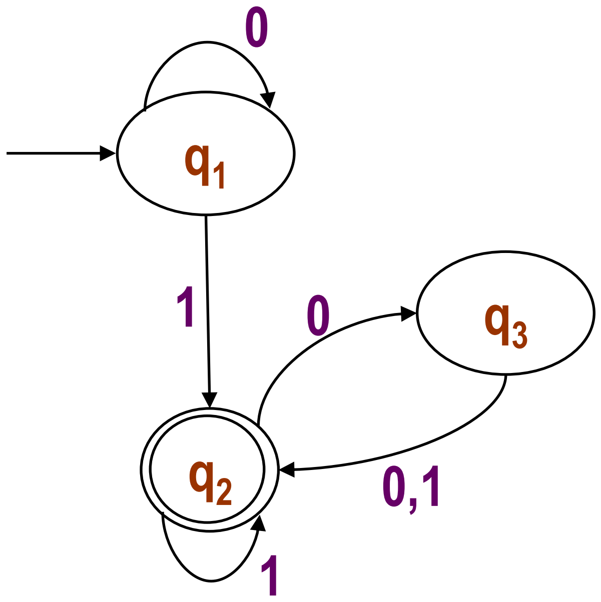
Example: $M = (Q, \Sigma, \delta, q_1, F)$, where

- $Q = \{q_1, q_2, q_3\}$,
- $\Sigma = \{0,1\}$,
- δ is described as
- q_1 is the start state
- $F = \{q_2\}$

Σ

Q

δ	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2



Acceptance/Recognition by DFA

Let M = (Q, ∑, δ, q_0, F) be a deterministic finite automaton and w = $w_1w_2…w_n$ be a string where each $w_i \in$ ∑. Then M accepts w if a sequence of states r_0, r_1, …, r_n in Q exists with three conditions:

- $r_0 = q_0$,
- $\delta(r_i, w_{i+1}) = r_{i+1}$, for i = 0, 1, …, n-1, and
- $r_n \in F$

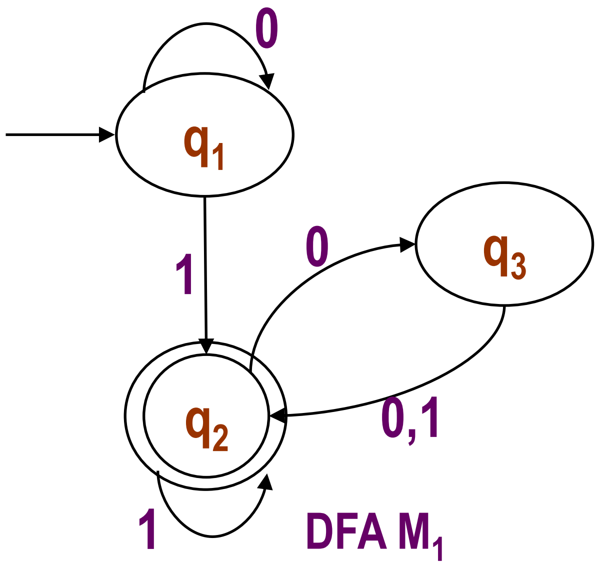
Therefore, M recognizes language A_M if A_M = {w | M accepts w}

Example:

$L(M_1) = A_{M1}$ (M_1 recognizes/accepts A_{M1}), where

A_{M1} = {w | w contains at least one 1 and

an even number of 0s follow the last 1}



DFA M_1

Non-deterministic Finite Automaton (NFA)

A *non-deterministic finite automaton (NFA)* is a 5-tuple *$(Q, \sum, \delta, q_0, F)$*, where

- *Q* is a finite set called the set of *states*,
- $\sum$ is a finite set called the *alphabet*,
- *$\delta: Q \times \sum_\varepsilon \rightarrow P(Q)$* is the *transition function*,
- *$q_0 \in Q$* is the *start state*, and
- *$F \subseteq Q$* is the *set of accept states (final states)*

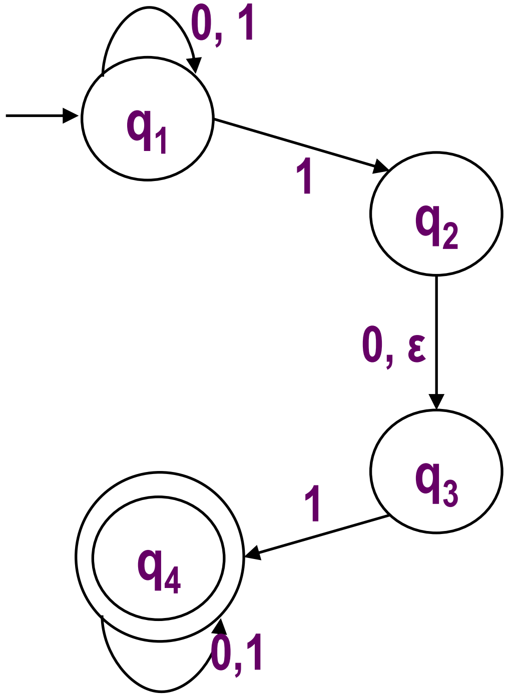
Example: *N* = *$(Q, \sum, \delta, q_1, F)$,* where

- *Q* = $\{q_1, q_2, q_3, q_4\}$,
- $\sum$ = {0,1},
- *δ* is described as
- *q_1* is the start state
- *F* = $\{q_4\}$

∑

Q

δ	0	1	ε
q_1	$\{q_1\}$	$\{q_1, q_2\}$	Φ
q_2	$\{q_3\}$	Φ	$\{q_3\}$
q_3	Φ	$\{q_4\}$	Φ
q_4	$\{q_4\}$	$\{q_4\}$	Φ



Acceptance/Recognition by NFA

Let N = (Q, ∑, δ, q_0, F) be a non-deterministic finite automaton and y = $y_1y_2\ldots y_n$ be a string where each $y_i \in \sum_\varepsilon$. Then N accepts y if a sequence of states $r_0, r_1, \ldots, r_m$ in Q exists with three conditions:

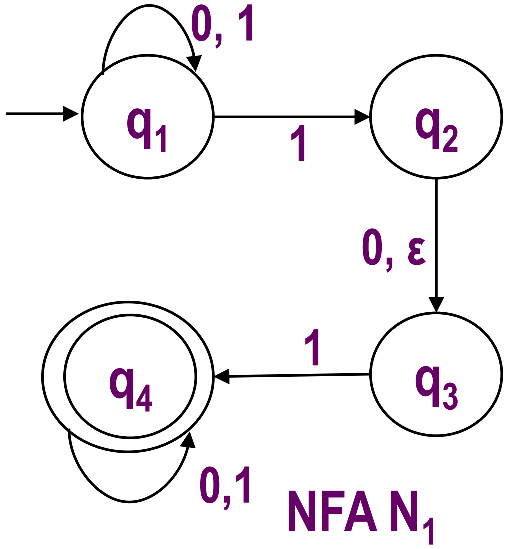
- $r_0 = q_0$,
- $r_{i+1} \in \delta(r_i, y_{i+1})$, for i = 0, 1, …, m-1, and
- $r_m \in F$

Therefore, N recognizes language A_N if A_N = {y | N accepts y}

Example:

$L(N_1)$ = *A_{N1}* (N_1 recognizes/accepts A_{N1}), where

A_{N1} = {y | y contains either 101 or 11 as a substring}



NFA N_1

Regular Operations

A language is called a regular language iff some DFA recognizes it

Let A and B be regular languages. The regular operations *union*, *concatenation* and *star* are defined as follows:

- Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- Concatenation: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$

Binary Operation

- Star: $A^* = \{x_1x_2\ldots x_k \mid k \geq 0 \text{ and } x_i \in A\}$

Unary Operation

Closure under Regular Operations

Closure Theorems:

- **The class of regular languages is closed under the union operation**

  (if A_1 and A_2 are regular languages, so is A_1 U A_2)

- **The class of regular languages is closed under the concatenation operation**

  (if A_1 and A_2 are regular languages, so is $A_1 \circ A_2$)

- **The class of regular languages is closed under the star operation**

  (if A is a regular language, so is A*)

Regular Expressions

R is a *regular expression* if R is

- a for some a in the alphabet ∑,
- ε,
- Φ,
- (R_1 U R_2), where R_1 and R_2 are regular expressions
- ($R_1 \circ R_2$), where R_1 and R_2 are regular expressions
- R_1*, where R_1 is a regular expression

Some Important Identities:

- $R^+ \equiv RR^*$ and $R^+ \; U \; \varepsilon \equiv R^*$
- $R \; U \; \Phi \equiv R$ and $R \circ \varepsilon \equiv R$
- *(R U ε)* may not equal *R* (Ex: if R = 0; then L(R) = {0}, but L(R U ε) = {0, ε})
- *(R ○ Φ)* may not equal *R* (Ex: if R = 0; then L(R) = {0}, but L(R ○ Φ) = Φ)

Example of Regular Expression

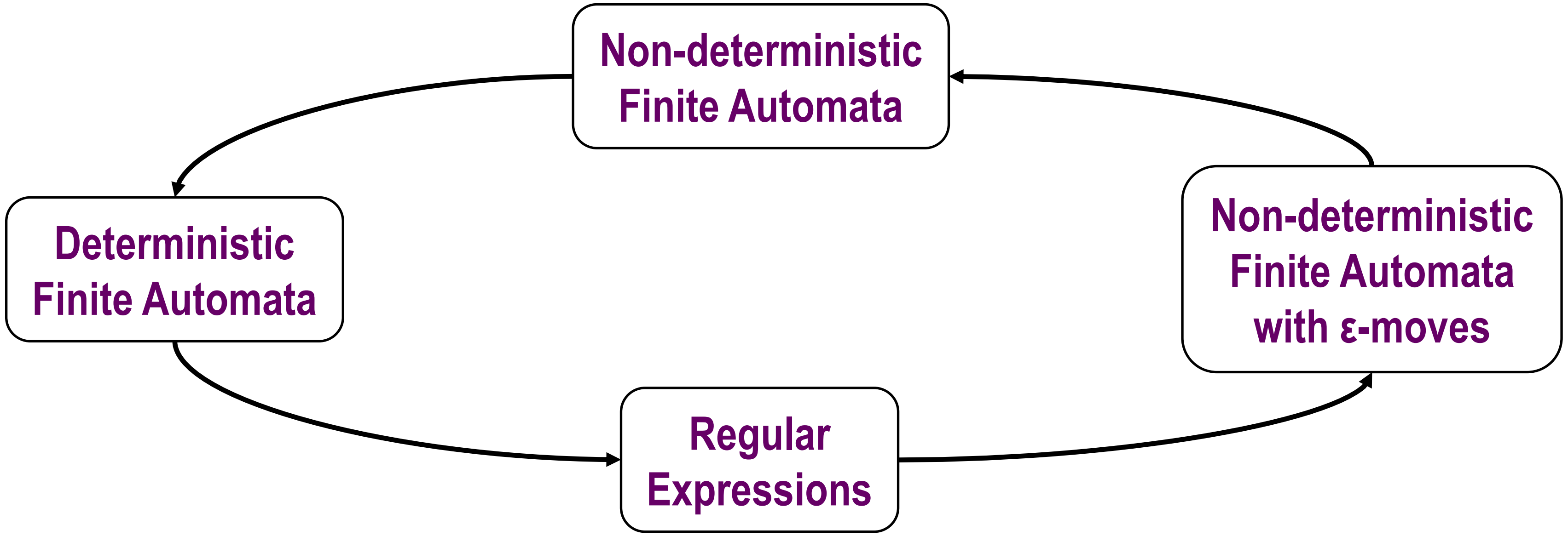
- Let D = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} is the alphabet of decimal digits; then a numerical constant that may include a fractional part and/or a sign may be described as a member of the language: (+ U – U ε) (D^+ U (D^+ . D^*) U (D^* . D^+))

Equivalence with Finite Automata

Two finite automata are *equivalent* if they accept the same regular language

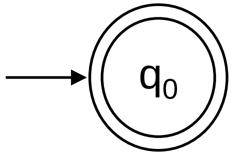
Theorems:

- Every non-deterministic finite automaton has an equivalent deterministic finite automaton
- A language is regular if and only if some non-deterministic finite automaton recognizes/accepts it
- A language is regular if and only if some regular expression describes it
- If a language L is accepted by a DFA, then L is denoted by a regular expression

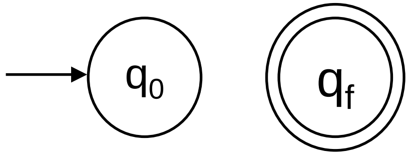

Non-deterministic Finite Automata
Deterministic Finite Automata
Non-deterministic Finite Automata with ε-moves
Regular Expressions

Regular Expression → NFA (with ε-moves)

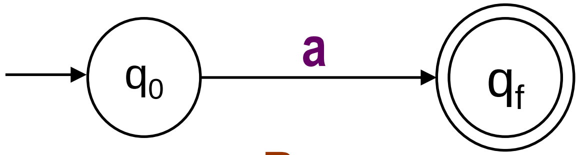
Let R be a regular expression. Then there exists an NFA with ε-transitions (M) that accepts L(R). The construction procedure is as follows:



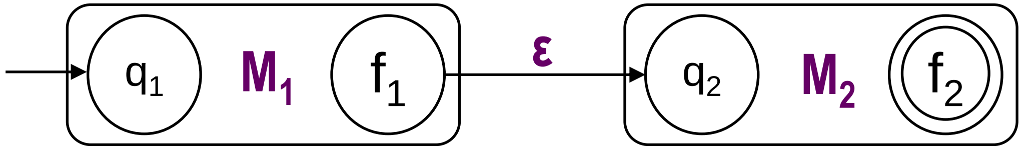
R = ε



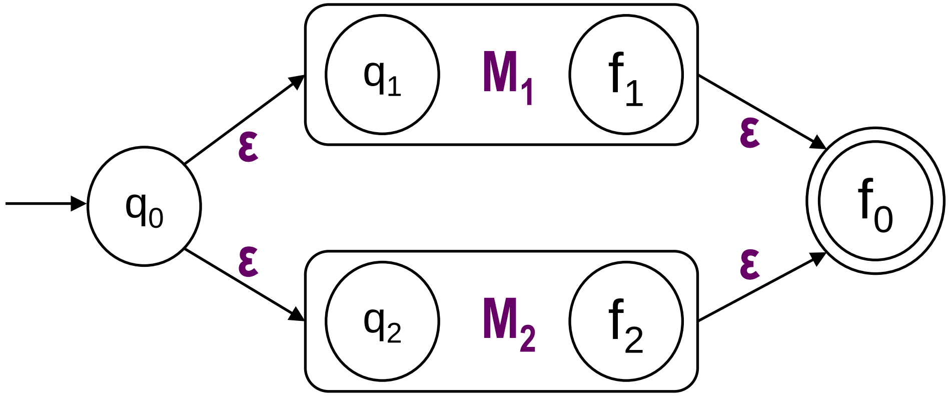
R = Φ



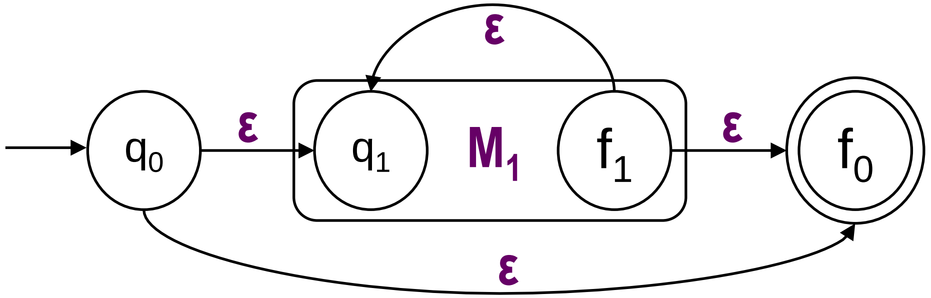
R = a



For Concatenation: $L(M) = L(M_1) \circ L(M_2)$



For Union: $L(M) = L(M_1) \cup L(M_2)$



For Star: $L(M) = L(M_1)^*$

Pumping Lemma: Proving Non-regularity

If *A* is a regular language, then there is a number *p* (the pumping length) where, *s* is any string in *A* of length at least *p*, then *s* may be divided into three pieces, *s* = *xyz* satisfying the following conditions:

- For each $i \geq 0$, $x y^i z \in A$
- $|y| > 0$, and
- $|xy| \leq p$

Examples:

- The following languages (denoted by B, C, D, E, F) are not regular:
  - $B = \{0^n 1^n \mid n \geq 0\}$
  - C = {w | w has an equal number of 0s and 1s}
  - $F = \{ww \mid w \in \{0, 1\}^*\}$
  - $D = \{1^n \mid n \geq 0\}$
  - $E = \{0^i 1^j \mid i > j\}$