Regular Languages

CS60001: Foundations of Computing Science

Pallab Dasgupta

Professor, Dept. of Computer Sc. & Engg.,
Indian Institute of Technology Kharagpur

Deterministic Finite Automaton (DFA)

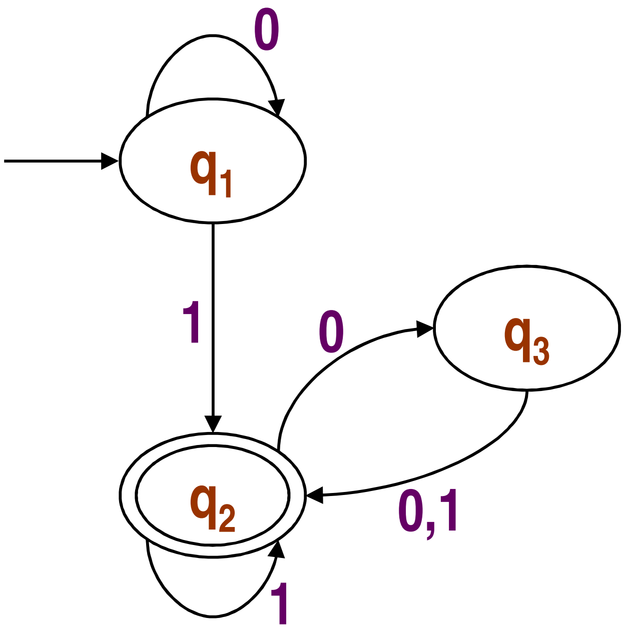
- A *deterministic finite automaton (DFA)* is a 5-tuple *(Q, ∑, δ, q_0, F)*, where
  - *Q* is a finite set called the *states*,
  - ∑ is a finite set called the *alphabet*,
  - *δ: Q × ∑ → Q* is the *transition function*,
  - *$q_0 \in Q$* is the *start state*, and
  - *$F \subseteq Q$* is the *set of accepted states (final states)*

- Example: *M = (Q, ∑, δ, q_1, F)*, where
  - *Q* = {q_1, q_2, q_3},
  - ∑ = {0,1},
  - δ is described as
  - *q_1* is the start state
  - *F* = {q_2}

∑

δ	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

Q



Acceptance/Recognition by DFA

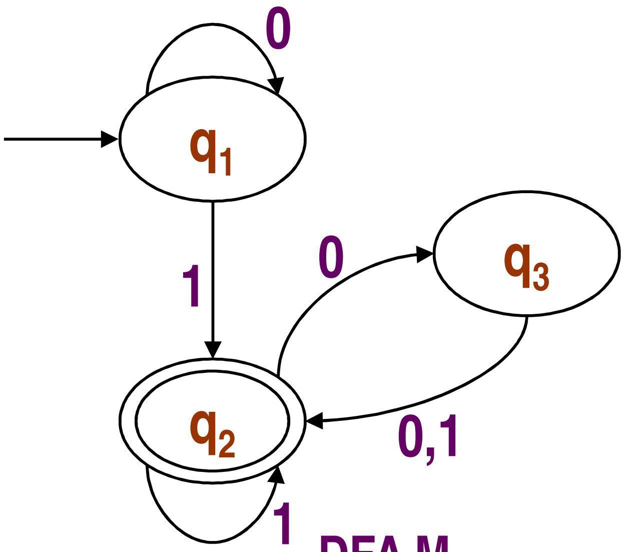
- ❑ Let $M = (Q, \sum, \delta, q_0, F)$ be a deterministic finite automaton and $w = w_1w_2\ldots w_n$ be a string where each $w_i \in \sum$. Then M accepts w if a sequence of states $r_0, r_1, \ldots, r_n$ in Q exists with three conditions:
  - $r_0 = q_0$,
  - $\delta(r_i,w_{i+1}) = r_{i+1}$, for i = 0, 1, …, n-1, and
  - $r_n \in F$

  Therefore, M recognizes language A_M if $A_M = \{w \mid M \text{ accepts } w\}$

- ❑ Example:

  $L(M_1) = A_{M1}$ (M_1 recognizes/accepts A_{M1}), where

  $A_{M1} = \{w \mid w$ contains at least one 1 and an even number of 0s follow the last 1$\}$



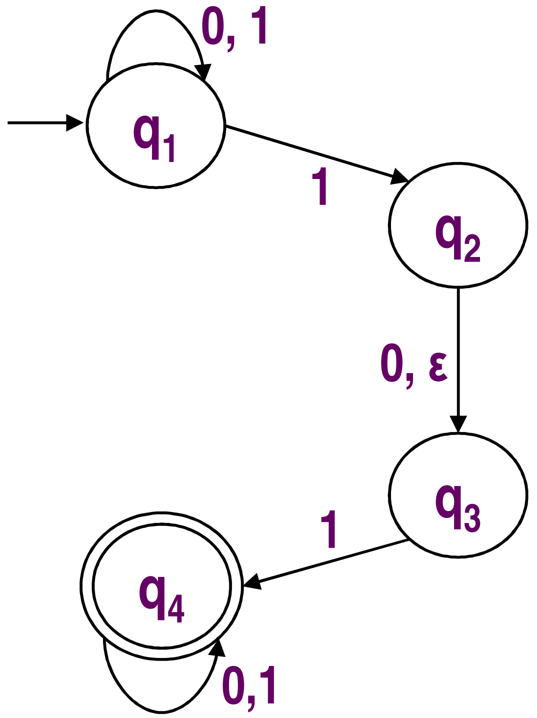
DFA M_1

Non-deterministic Finite Automaton (NFA)

- A *non-deterministic finite automaton (NFA)* is a 5-tuple *(Q, $\sum$, δ, q_0, F)*, where
  - *Q* is a finite set called the *states*,
  - $\sum$ is a finite set called the *alphabet*,
  - *$\delta: Q \times \sum_\varepsilon \rightarrow P(Q)$* is the *transition function*,
  - *$q_0 \in Q$* is the *start state*, and
  - *$F \subseteq Q$* is the *set of accepted states (final states)*

- Example: *$N = (Q, \sum, \delta, q_1, F)$*, where
  - *Q* = {q_1, q_2, q_3, q_4},
  - $\sum$ = {0,1},
  - δ is described as
  - *q_1* is the start state
  - *F* = {q_4}

		$\sum$		
	δ	0	1	ε
Q	q_1	{q_1}	{q_1, q_2}	Φ
	q_2	{q_3}	Φ	{q_3}
	q_3	Φ	{q_4}	Φ
	q_4	{q_4}	{q_4}	Φ



Acceptance/Recognition by NFA

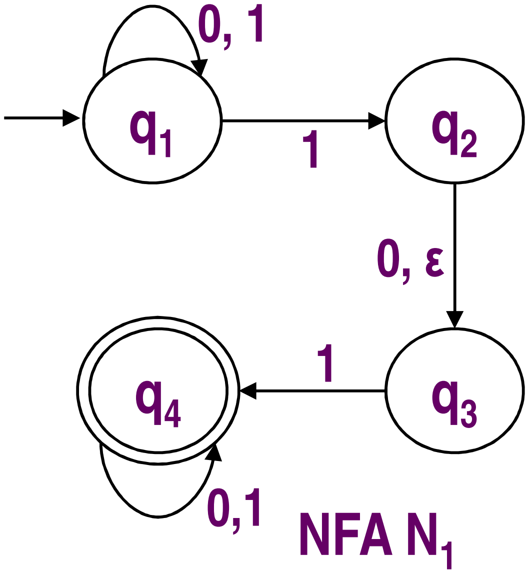
- Let N = (Q, ∑, δ, q_0, F) be a non-deterministic finite automaton and y = $y_1y_2…y_n$ be a string where each $y_i \in \sum_\varepsilon$. Then N accepts y if a sequence of states r_0, r_1, …, r_m in Q exists with three conditions:
  - $r_0 = q_0$,
  - $r_{i+1} \in \delta(r_i, y_{i+1})$, for i = 0, 1, …, m-1, and
  - $r_m \in F$

  Therefore, N recognizes language A_N if A_N = {y | N accepts y}

- Example:

  $L(N_1) = A_{N1}$ (N_1 recognizes/accepts A_{N1}), where

  A_{N1} = {y | y contains either 101 or 11 as a substring}



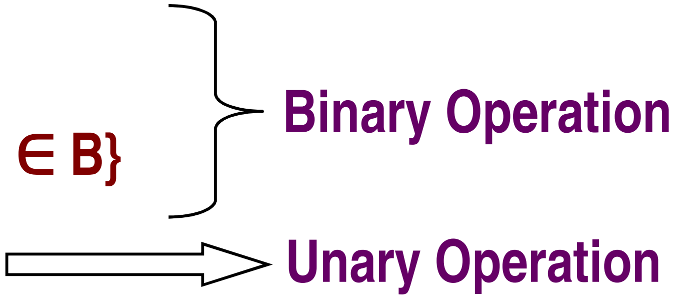
NFA N_1

Regular Operations

- A language is called a **regular language** if some automaton recognizes it

- Let A and B be regular languages. The regular operations *union*, *concatenation* and *star* are defined as follows:

  - Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
  - Concatenation: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$
  - Star: $A^* = \{x_1x_2\ldots x_k \mid k \geq 0 \text{ and } x_i \in A\}$



Binary Operation

Unary Operation

Closure under Regular Operations

- ❑ **Closure Theorems:**

  - The class of regular languages is closed under the union operation

    (if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$)

  - The class of regular languages is closed under the concatenation operation

    (if A_1 and A_2 are regular languages, so is $A_1 \circ A_2$)

  - The class of regular languages is closed under the star operation

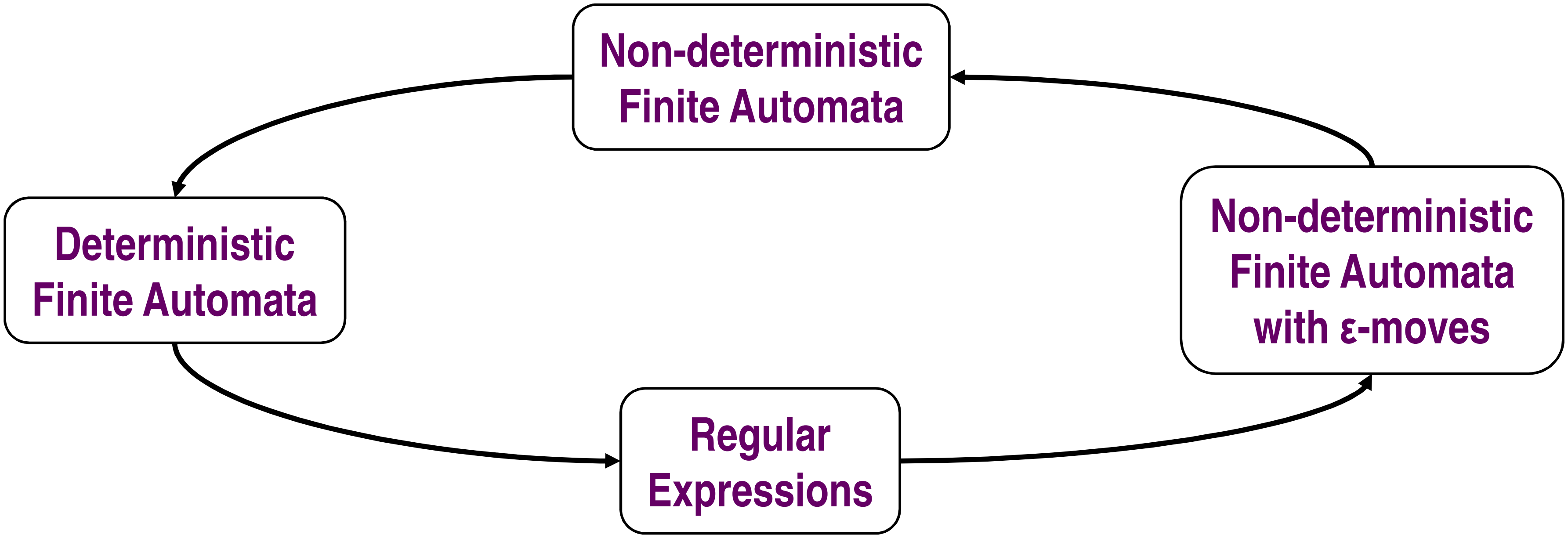
    (if A is a regular language, so is A*)

Regular Expressions

- R is a *regular expression* if R is
  - a for some a in the alphabet ∑,
  - ε,
  - Φ,
  - $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions
  - $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions
  - $R_1{}^*$, where R_1 is a regular expression
- Some Important Identities:
  - $R^+ \equiv RR^*$ and $R^+ \cup \varepsilon \equiv R^*$
  - $R \cup \Phi \equiv R$ and $R \circ \varepsilon \equiv R$
  - $(R \cup \varepsilon)$ may not equal R (Ex: if R = 0; then L(R) = {0}, but L(R U ε) = {0, ε})
  - $(R \circ \Phi)$ may not equal R (Ex: if R = 0; then L(R) = {0}, but L(R ○ Φ) = Φ)
- Example of Regular Expression
  - Let D = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} is the alphabet of decimal digits; then a numerical constant that may include a fractional part and/or a sign may be described as a member of the language: $(+ \cup - \cup \varepsilon)\ (D^+ \cup D^+ . D^* \cup D^* . D^+)$

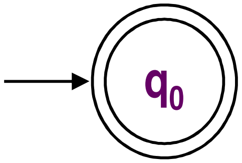
Equivalence with Finite Automata

- Two finite automata are *equivalent* if they accept the same regular language
- Theorems:
  - Every non-deterministic finite automaton has an equivalent deterministic finite automaton
  - A language is regular if and only if some non-deterministic finite automaton recognizes/accepts it
  - A language is regular if and only if some regular expression describes it
  - If a language L is accepted by a DFA, then L is denoted by a regular expression

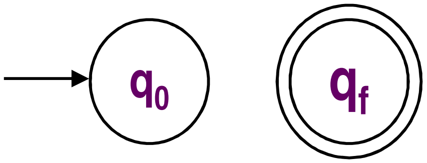

Non-deterministic Finite Automata
Deterministic Finite Automata
Non-deterministic Finite Automata with ε-moves
Regular Expressions

Regular Expression → NFA (with ε-moves)

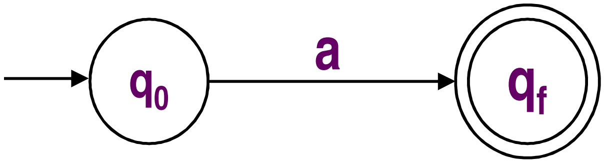
- ❑ Let r be a regular expression. Then there exists an NFA with ε-transitions (M) that accepts L(r). The construction procedure is as follows:



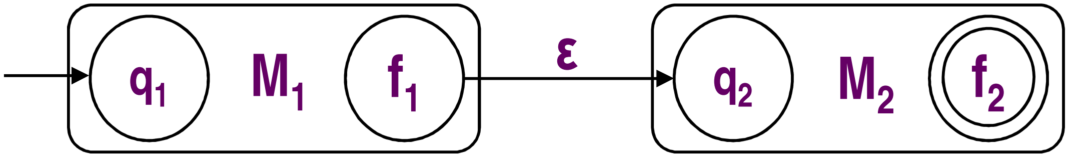
r = ε



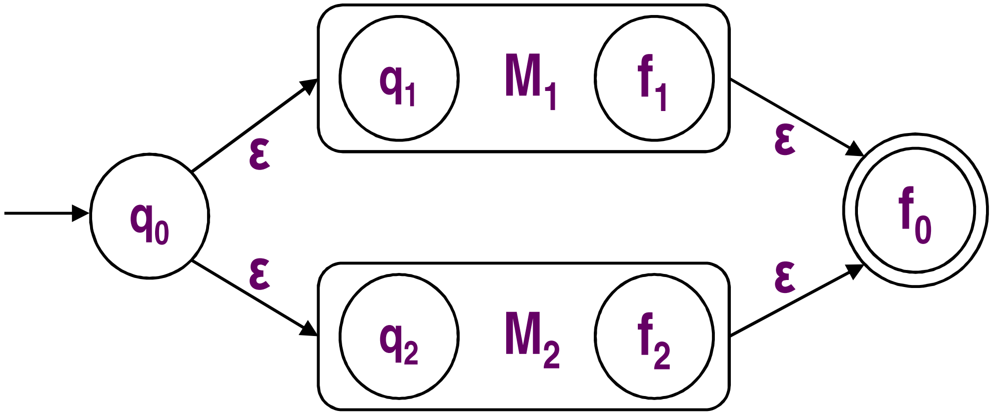
r = Φ



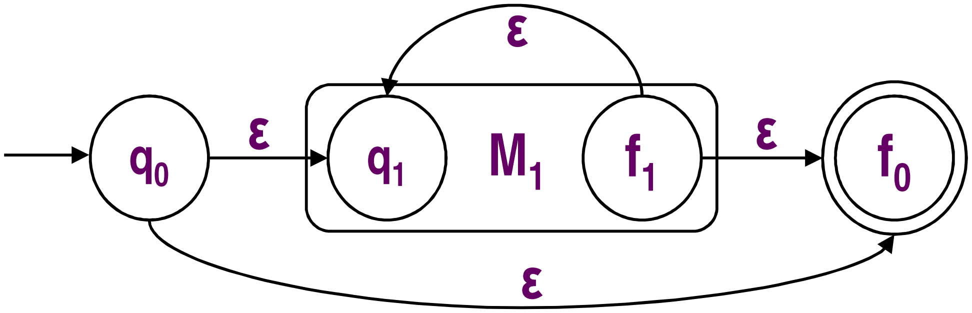
r = a



For Concatenation: $L(M) = L(M_1) \circ L(M_2)$



For Union: $L(M) = L(M_1)\ U\ L(M_2)$



For Star: $L(M) = L(M_1)^*$

Pumping Lemma: Proving Non-regularity

- If *A* is a regular language, then there is a number *p* (the pumping length) where, *s* is any string in *A* of length at least *p*, then *s* may be divided into three pieces, *s = xyz* satisfying the following conditions:
  - For each $i \geq 0$, $xy^iz \in A$
  - $|y| > 0$, and
  - $|xy| \leq p$

- Examples:
  - The following languages (denoted by B, C, D, E, F) are not regular:
    - $B = \{0^n1^n \mid n \geq 0\}$
    - C = {w | w has an equal number of 0s and 1s}
    - $F = \{ww \mid w \in \{0, 1\}^*\}$
    - $D = \{1^{n^2} \mid n \geq 0\}$
    - $E = \{0^i1^j \mid i > j\}$