Practice Problems on Martingales

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1. Alice rolls a fair six-sided die (that is, she randomly sample numbers i.i.d. uniformly from $\{1, 2, 3, 4, 5, 6\}$), until she rolls an odd number. What is the expected number of 5's that she rolls?

2. Consider a gambling game in which a player first rolls one standard die. If the outcome of the roll is X then she rolls X new standard dice and her gain Z is the sum of the outcome of the X dice. What is the expected gain of the gambler?

3. Suppose there are n servers sharing a communication channel. At each time step, each server sends one message with probability $\frac{1}{n}$. Message transmission is successful in a time step if only one server sends in that time step. What is the expected number of time steps until all servers have sent at least one message. [Hint: use Wald's equation]

4. Let $X = (X_1, \ldots, X_n)$ be a sequence of n characters, each drawn independently and uniformly from an alphabet of size s and $B = (B_1, \ldots, B_k)$ any fixed string of size k drawn from the same alphabet. What is the expected number of occurences of B in X? Derive concentration bound around of the number of occurences around the mean. [Hint: Doob martingale and Azuma's inequality]

5. Suppose we throw m balls into n bins independently, uniformly at random. What is the expected number of empty bins? How that number is contentrated around its mean? [Hint: Doob martingale and Azuma's inequality]