# Practice Problems on Markov Chain

Palash Dey
Indian Institute of Technology, Kharagpur

September 6, 2023

1. Suppose we have a Monte Carlo randomized algorithm $\mathcal{A}$ for some decision problem $\Pi$ which, on every input $x$, outputs correct answer with probability at least $3/4$. Suppose $\mathcal{A}$ uses $\mathcal{O}(\log n)$ random bits where $n$ is the size of input. Prove that there exists a deterministic polynomial time algorithm for the problem $\Pi$.

2. Let $\mathcal{P}$ and $\mathcal{Q}$ be two distribution over the $\sigma$-algebra $([n], 2^{[n]})$. Then prove the following.
$$2d_{TV}(\mathcal{P}, \mathcal{Q}) = ||\mathcal{P} - \mathcal{Q}||_1$$

3. Consider drunken walk on integers 0 to $n$ where the transition probability from $k$ to $k+1$ is $\frac{1}{3}$ for every $k \in [n-1]$, the transition probability from $k$ to $k-1$ is $\frac{2}{3}$ for every $k \in [n]$, the transition probability from 0 to 1 is 1, and $n$ is an absorbing state. Prove that the expected number of steps to reach $n$ from 0 is $\Omega(2^n)$. Can you give an example of a 3SAT formula where the 2SAT style randomized algorithm indeed takes $\Omega(2^n)$ steps? Hint: Write down a 3SAT formula with exactly one satisfying assignment.

4. A random walk on a connected undirected graph is aperiodic if and only if the graph is not bipartite.

5. Show that the mixing time of a random walk on an $n$ dimensional hypercube is at most $n \ln n + n \ln(1/\varepsilon)$.

6. Consider the following card shuffling: insert the top card at a uniformly random position from 1 to $n$. Model the shuffling process as a Markov chain. Show that the uniform distribution is the unique stationary distribution. Show that the mixing time is at most $n \ln n + n \ln(1/\varepsilon)$.