Indian Institute of Technology Kharagpur

CS60029 Randomized Algorithm Design – Class Test 1

Total marks: 30 Duration: 1 hour

Name: ______________________________

Roll Number: ______________________________

Answer all questions in the question paper itself. Keep your answers brief and precise.

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1. (a) (3 points) Suppose an undirected unweighted graph has k min-cuts. Calculate the probability that one round of Karger's randomized algorithm outputs a min-cut.

   (b) (7 points) Prove or disprove: Karger's algorithm for computing a min-cut in an undirected graph also works ditto for weighted graphs.

Solution:

(a) $\frac{2k}{n(n-1)}$

(b) Assume all weights are positive. Sample edge with probability proportional to its weight.

2. (10 points) Let k be even and let X be a random variable for which $\mu_X^k = \mathbb{E}\left[(X-\mu_X)^k\right]$ exists. Show that

$$\Pr\left[|X-\mu_X| > t\sqrt[k]{\mu_X^k}\right] \leqslant \frac{1}{t^k}.$$

Explain, in not more than 2 lines, why it is difficult to derive a similar inequality when k is odd.

Solution: We have

$$\begin{aligned}
\Pr\left[|X-\mu_X| > t\sqrt[k]{\mu_X^k}\right] &= \Pr\left[|X-\mu_X|^k > t^k\mu_X^k\right] \\
&= \Pr\left[(X-\mu_X)^k > t^k\mu_X^k\right] \\
&\leqslant \frac{\mathbb{E}\left[(X-\mu_X)^k\right]}{t^k\mu_X^k} \\
&= \frac{1}{t^k}
\end{aligned}$$

where the second equality follows from k being even and the inequality follows is obtained by applying Markov's inequality on the random variable $(X-\mu_X)^k$ which takes only non-negative values as k is even.

For odd values of k, the random variable $(X-\mu_X)^k$ is not guaranteed to take only non-negative values and so Markov's inequality cannot be used to derive a similar bound.

3. (10 points) Let $X_1, X_2, \ldots, X_n$ be n integers chosen independently and uniformly at random from the set $\{0, 1, 2\}$. Let $X = \sum_{i=1}^{n} X_i$ and $0 < \delta < 1$. Derive a Chernoff bound for $\Pr[X \geqslant (1+\delta)n]$.

Solution: Define $Y_i = X_i - 1$. Let $Y = \sum_{i=1}^{n} Y_i$. Then $Y = X - n$. Also, Y_i's are independently and uniformly distributed over $\{-1, 0, 1\}$. We first derive a bound on $\Pr[Y > n\delta]$. Let $t \in \mathbb{R}^+$.

$$
\begin{aligned}
\Pr[Y > n\delta] &= \Pr[e^{tY} > e^{tn\delta}] \\
&\leqslant \frac{\mathbb{E}\left[e^{tY}\right]}{e^{tn\delta}} \\
&= \frac{\prod_{i=1}^{n} \mathbb{E}\left[e^{tY_i}\right]}{e^{tn\delta}} \\
&= \frac{(e^{-t} + 1 + e^{t})^n}{3^n e^{tn\delta}} \\
&= \frac{\left(1 + 2\sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}\right)^n}{3^n e^{tn\delta}} \\
&\leqslant \frac{\left(3\sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}\right)^n}{3^n e^{tn\delta}} \\
&\leqslant \frac{\left(\sum_{k=0}^{\infty} \frac{t^{2k}}{2^k k!}\right)^n}{e^{tn\delta}} \\
&= \frac{\left(e^{t^2/2}\right)^n}{e^{tn\delta}}
\end{aligned}
$$

Note that $e^{t^2 n/2 - tn\delta}$ is minimised for $t = \delta$. Substituting $t = \delta$ in the expression, we get

$$\Pr[Y > n\delta] \leqslant e^{-\delta^2 n/2}$$

It now follows that

$$\Pr[X > (1+\delta)n] = \Pr[X - n > n\delta] = \Pr[Y > n\delta] \leqslant e^{-\delta^2 n/2}$$

——— Space for Rough Work ———