CS60007 Algorithm Design and Analysis 2018
Supplementary for Lecture 6

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August 1, 2018

Let $\mathcal{G}$ be a weighted and undirected[1] graph with weight function $w : \mathcal{E}[\mathcal{G}] \longrightarrow \mathbb{R}$. We define $d(u, v)$ to be the distance from u to v in $\mathcal{G}$ for $u, v \in \mathcal{V}[\mathcal{G}]$. Let $\ell(v, i)$ be the label of vertex $v \in \mathcal{V}$ at the beginning of the i-th iteration in the Dijkstra's algorithm and $d(v) = \ell(v, i)$ if the Dequeue operation returns vertex v in the iteration i. We have the following property about $\ell(v, i)$. Let $s \in \mathcal{V}[\mathcal{G}]$ be the vertex from which we need to find the distance to every other vertices in $\mathcal{G}$. Then we have the following result.

Lemma 1. *For every* $i \in [n]$ *and* $v \in \mathcal{V}[\mathcal{G}]$, *we have* $d(s, v) \leqslant \ell(v, i)$.

Proof. We will prove it by induction on i. For $i = 1$, the result follows from the observation that $v_1 = s, \ell(s, 1) = 0, \ell(v, 1) = \infty$ for every $v \in \mathcal{V}[\mathcal{G}] \setminus \{s\}$.

Let us now assume the statement for i and prove it for $i + 1$. The results holds for every vertex in $\mathcal{V}' = \{v \in \mathcal{V}[\mathcal{G}] : \ell(v, i) = \ell(v, i + 1)\}$ from Induction hypothesis. Hence, if $\mathcal{V}[\mathcal{G}] = \mathcal{V}'$, then we have nothing to prove. So let us assume that $\mathcal{V}[\mathcal{G}] \neq \mathcal{V}'$. Let $w \in \mathcal{V}[\mathcal{G}] \setminus \mathcal{V}'$ be any vertex in $\mathcal{V}[\mathcal{G}] \setminus \mathcal{V}'$. Then we have $\ell(w, i + 1) = \ell(x, i) + w(\{x, w\})$ for some $x \in \mathcal{V}[\mathcal{G}] \setminus \{w\}$. Now we have the following:

$$d(s, w) \leqslant d(s, x) + w(\{x, w\}) \leqslant \ell(x, i) + w(\{x, w\}) = \ell(w, i + 1)$$

The first inequality follows from the definition of distance and the second inequality follows from the Induction hypothesis for the vertex x. This concludes the proof of the lemma. $\square$

[1] All the results claimed here extend to directed graphs too with analogous argument. We choose to restrict ourselves to undirected graphs only for simplicity.

Using Lemma 1, we now prove the correctness of the Dijkstra's algorithm. Let v_i be the vertex which Dequeue returns in the i-th iteration of the loop in the Dijkstra's algorithm.

Theorem 2. *If all the edge weights of the graph $\mathcal{G}$ are positive, then we have $d(v_i) = d(s, v_i)$ for every $i \in [n]$ which is the same as saying that $d(v) = d(s, v)$ for every $v \in \mathcal{V}[\mathcal{G}]$.*

Proof. We can assume without loss of generality that every vertex is reachable from s. If not, then let $\mathcal{V}' \subseteq \mathcal{V}[\mathcal{G}]$ be the set of vertices in $\mathcal{G}$ which are unreachable from s. We have $d(s, v) = \infty$ for every $v \in \mathcal{V}'$ (because there is no path from s to v in $\mathcal{G}$) and the result follows form Lemma 1. So we can restrict our attention only to the vertices that are reachable from s in $\mathcal{G}$.

We observe that $d(v_i) = \ell(v_i, i)$. Hence, we have $d(v_i) \geqslant d(s, v_i)$ due to Lemma 1.

We now prove that $d(v_i) \leqslant d(s, v_i)$ for every $i \in [n]$ using Induction on i. The base case follows immediately from the observation that $v_1 = s$ and we have $d(v_1) = 0 \leqslant d(s, v_1)$ since all the edge weights are positive.

Let us now assume the statement for i and prove it for $i + 1$. If $\ell(v_{i+1}, i) = \infty$, then let $\mathcal{W} = \{v \in \mathcal{V}[\mathcal{G}] : \ell(v, i) = \infty\}$. Clearly, $\mathcal{W} \neq \emptyset$ since $v_{i+1} \in \mathcal{W}$ by our assumption. However, this implies that there is no edge in the cut $(\mathcal{W}, \mathcal{V}[\mathcal{G}] \setminus \mathcal{W})$. In particular every vertex in $\mathcal{W}$ is unreachable from s in $\mathcal{G}$ which contradicts our assumption that $\mathcal{G}$ is connected. So, we have $\ell(v_{i+1}, i) < \infty$. Let $\mathfrak{p} = (u_0(= s), u_1, u_1, \dots, u_k(= v_{i+1}))$ be a shortest path from s to v_{i+1} in $\mathcal{G}$. We claim that $u_j \in \{v_1, v_2, \dots, v_i\}$ for every $0 \leqslant j \leqslant k - 1$. Suppose not, then there exists a vertex u_t with $0 \leqslant t \leqslant k - 1$ such that $u_t \notin \{v_1, v_2, \dots, v_i\}$. We observe the following chain of inequalities:

$$\ell(u_t, i) \leqslant \ell(u_{t-1}, i) + w(\{u_{t-1}, u_t\}) = d(u_{t-1}) + w(\{u_{t-1}, u_t\}) < d(s, v_{i+1}) \leqslant \ell(v_{i+1}, i)$$

The first inequality follows from the update procedure (which is the same as 'relaxation' in the CLRS book) of the algorithm, the equality follows from the fact that $u_{t-1} \in \{v_1, v_2, \dots, v_i\}$, the third inequality follows from the assumption that the weight of every edge in $\mathcal{G}$ is positive, and the third inequality follows from Lemma 1. However it contradicts our choice of v_{i+1} since $\ell(u_t, i) < \ell(v_{i+1}, i)$ and $u_t, v_{i+1} \notin \{v_1, v_2, \dots, v_i\}$. Hence we have $u_j \in \{v_1, v_2, \dots, v_i\}$ for every $0 \leqslant j \leqslant k - 1$. Since, ℓ values are always decreased when it is updated for any vertex, we have $\ell(v_{i+1}, i) \leqslant d(s, v_{i+1})$.

Hence, we have $d(v_i) \leqslant d(s, v_i)$ for every $i \in [n]$ which concludes the proof of correctness of the Dijkstra's algorithm. □

Can you find the places in the proof of Theorem 2 where we have used the fact that the weight of every edge in the input graph is positive? Can you find an example with one negative weight edge and no negative weight cycle where Dijkstra's algorithm fails?