Computer Science & Engineering Department
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Theory of Computation: CS41001 (*Class Test I*)

From 20:00 - 20:45 hrs *Date : 20^{th} August, 2014*

Roll No.: **Name:**

Marks out of 20:

1. Reduce the following λ-expression to β-normal form. Rename bound variables if necessary.

$$(\lambda xy \cdot (\lambda xy \cdot xy)(\lambda x \cdot x)x)(\lambda xy \cdot y).$$

[2]

<u>*Sol.:*</u>

$$\begin{array}{ll} & (\lambda xy \cdot (\lambda xy \cdot xy)(\lambda x \cdot x)x)(\lambda xy \cdot y) \\ \to_\beta & \lambda y \cdot (\lambda xy \cdot xy)(\lambda x \cdot x)(\lambda xy \cdot y) \\ \to_\beta & \lambda y \cdot (\lambda y \cdot (\lambda x \cdot x)y)(\lambda xy \cdot y) \\ \to_\beta & \lambda y \cdot (\lambda x \cdot x)(\lambda xy \cdot y) \ \text{(after renaming)} \\ \to_\beta & \lambda y \cdot (\lambda xy \cdot y) \\ \to_\beta & \lambda zxy \cdot y \ \text{(after renaming)} \end{array}$$

2. Show that $K\Omega I$ does not have a normal form, where $K \equiv \lambda xy \cdot x$, $I \equiv \lambda x \cdot x$, $\Omega \equiv (\omega\omega)$, where $\omega \equiv \lambda x \cdot xx$. **[2]**

<u>*Sol.:*</u>

$$\begin{array}{ll} & K\Omega I \\ \equiv & (\lambda xy \cdot x)\Omega I \\ \to_\beta & (\lambda y \cdot \Omega)I \\ \to_\beta & \Omega \\ \equiv & (\lambda x \cdot xx)(\lambda x \cdot xx) \\ \to_\beta & (\lambda x \cdot xx)(\lambda x \cdot xx) \\ \cdots & \Omega \end{array}$$

3. A closed λ-term M is called a *fixed-point combinator* if for all λ-term F, $MF =_\beta F(MF)$.

   (a) Give an example (with proof) of a *fixed-point combinator*.

   (b) Construct a *fixed-point combinator* using the λ-term $A \equiv \lambda tor \cdot r\,(o\,t\,o\,r)$.

[2 + 2]

<u>*Sol.:*</u>

(a) The Y combinator is defined as follows:
$Y \equiv \lambda f \cdot (\lambda x \cdot f(xx))(\lambda x \cdot f(xx))$. Let F be any λ-term,

$$\begin{aligned}
& YF \\
\equiv\ & (\lambda f \cdot (\lambda x \cdot f(xx))(\lambda x \cdot f(xx)))F \\
\to_\beta\ & (\lambda x \cdot F(xx))(\lambda x \cdot F(xx)) \\
\to_\beta\ & F((\lambda x \cdot F(xx))(\lambda x \cdot F(xx))) \\
\to_\beta\ & F(YF).
\end{aligned}$$

(b) Define $B = AAA$. We claim that for all λ-term F, $BF =_\beta F(BF)$.

$$\begin{aligned}
& BF \\
\equiv\ & AAAF \\
\equiv\ & (\lambda tor \cdot r(otor)AAF, \\
\to_\beta\ & (\lambda or \cdot r(oAor)AF, \\
\to_\beta\ & (\lambda r \cdot r(AAAr)F, \\
\to_\beta\ & F(AAAF) \\
\equiv\ & F(BF).
\end{aligned}$$

4. The *Church numerals* are defined as $c_n = \lambda fx \cdot f^n(x)$, for $n = 0, 1, \cdots$, where $f^0(x) \equiv x$ and $f^{n+1}(x) \equiv f(f^n(x))$. Show that $A\, c_m\, c_n = c_{m+n}$, where $A \equiv \lambda xypq \cdot xp(ypq)$. **[3]**

   <u>*Sol.:*</u>

$$
\begin{aligned}
& A_+ c_m c_n \\
\equiv\ & (\lambda xypq \cdot xp(ypq)) c_m c_n \\
\to_\beta\ & \lambda pq \cdot c_m p (c_n pq) \\
\to_\beta\ & \lambda pq \cdot (\lambda fx \cdot f^m(x)) p ((\lambda fx \cdot f^n(x)) pq) \\
\to_\beta\ & \lambda pq \cdot (\lambda x \cdot p^m(x)) ((\lambda fx \cdot f^n(x)) pq) \\
\to_\beta\ & \lambda pq \cdot (\lambda x \cdot p^m(x)) ((\lambda x \cdot p^n(x)) q) \\
\to_\beta\ & \lambda pq \cdot (\lambda x \cdot p^m(x)) (p^n(q)) \\
\to_\beta\ & \lambda pq \cdot p^m(p^n(q)) \\
\to_\beta\ & \lambda pq \cdot p^{m+n}(q) \\
\equiv\ & c_{m+n}.
\end{aligned}
$$

5. We define *true* $\equiv T \equiv K \equiv \lambda xy \cdot x$, *false* $\equiv F \equiv K_* \equiv \lambda xy \cdot y$, an ordered pair $[M, N] \equiv \lambda x \cdot xMN$, and an ordered n-*tuple* $[M_1, \cdots, M_n] \equiv [M_1, [M_2, \cdots, M_n]]$ for $n \geq 3$.
Define π_4^2 that takes a 4-*tuple* (λ-term) and projects the second component i.e. $\pi_4^2\,[M_1, M_2, M_3, M_4] \to_\beta^* M_2$. Demonstrate its reduction. Also define π_4^4. **[3+1]**
Note: There was an error in the question $[M_1, M_2, M_2, M_3]$ will be $[M_1, M_2, M_3, M_4]$.

<u>*Sol.:*</u> We define $\pi_4^2 \equiv \lambda x \cdot xFT$. Also, $[M_1, M_2, M_3, M_4] \equiv \lambda x_1 \cdot x_1 M_1(\lambda x_2 \cdot x_2 M_2(\lambda x_3 \cdot x_3 M_3 M_4))$. Following is the reduction:

$$
\begin{aligned}
& (\lambda x \cdot xFT)(\lambda x_1 \cdot x_1 M_1(\lambda x_2 \cdot x_2 M_2(\lambda x_3 \cdot x_3 M_3 M_4))), \\
\to_\beta\ & (\lambda x_1 \cdot x_1 M_1(\lambda x_2 \cdot x_2 M_2(\lambda x_3 \cdot x_3 M_3 M_4)))FT \\
\to_\beta\ & (FM_1(\lambda x_2 \cdot x_2 M_2(\lambda x_3 \cdot x_3 M_3 M_4)))T \\
\to_\beta^*\ & (\lambda x_2 \cdot x_2 M_2(\lambda x_3 \cdot x_3 M_3 M_4))T \\
\to_\beta\ & TM_2(\lambda x_3 \cdot x_3 M_3 M_4) \\
\to_\beta^*\ & M_2.
\end{aligned}
$$

$\pi_4^4 \equiv \lambda x \cdot xFFFT$.

6. Let $\overline{n}, n = 0, 1, 2, \cdots$, be λ-numerals in β-normal form. Assume the existence of the following λ-terms corresponding to this numerals:

- Z - test for $\overline{0}$ i.e. $Z\,\overline{0} =_\beta T$ and $Z\,\overline{n} =_\beta F$, for $n > 0$.
- S - successor function i.e. $S\,\overline{n} =_\beta \overline{n+1}$.
- P - predecessor function i.e. $P\,\overline{0} =_\beta \overline{0}$ and $P\,\overline{n} =_\beta \overline{n-1}$, for $n > 0$.

We define the following λ-term:

$$G = \lambda fmn \cdot (Z\,m)\,n\,(S\,(f\,(P\,m)\,n)).$$

(a) Show that $(Y\,G)\,\overline{2}\,\overline{3} \rightarrow^*_\beta \overline{5}$. What is your conclusion?

(b) Design a λ-term G that can be used to multiply two λ-numerals.

[3+2]

Note: There was an error - F was used for λ-term *false* and also for the name of the λ-expression $\lambda fmn \cdot (Z\,m) \cdots$.

Sol.:

(a) Following is the reduction:

$$\begin{aligned}
& (Y\,G)\,\overline{2}\,\overline{3} \\
\rightarrow^*_\beta\; & G(Y\,G)\,\overline{2}\,\overline{3} \\
\rightarrow^*_\beta\; & (\lambda mn \cdot (Z\,m)\,n\,(S\,((Y\,G)\,(P\,m)\,n)))\,\overline{2}\,\overline{3} \\
\rightarrow^*_\beta\; & (Z\,\overline{2})\,\overline{3}\,(S\,((Y\,G)\,(P\,\overline{2})\,\overline{3})) \\
\rightarrow^*_\beta\; & (S\,((Y\,G)\,(P\,\overline{2})\,\overline{3})) \\
\rightarrow^*_\beta\; & (S\,((Y\,G)\,\overline{1}\,\overline{3})) \\
\rightarrow^*_\beta\; & (S\,(S\,((Y\,G)\,\overline{0}\,\overline{3}))) \\
\rightarrow^*_\beta\; & (S\,(S\,((Z\,\overline{0})\,\overline{3}\,(S\,((Y\,G)\,(P\,\overline{0})\,\overline{3}))))) \\
\rightarrow^*_\beta\; & (S\,(S\,\overline{3})) \\
\rightarrow^*_\beta\; & \overline{5}.
\end{aligned}$$

(b) The λ-term for multiplication is

$$G = \lambda fmn \cdot (Z\,m)\,\overline{0}\,(Plus\,(f\,(P\,m)\,n)),$$

where $Plus$ is YG.