# Pre RMO Workshop (22nd August-19th September, 2013)

# INEQUALITIES

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**Note**: *The following problems are of mixed difficulty levels. Most of the problems are taken from various mathematics competitions from around the world including our very own RMO, and INMO. Although the problems appear to be difficult, most of these problems have very short, and elegant solutions, which is the beauty of Olympiad level problems. So, if you get stuck on some problem, do not give up, but try a different approach. Even after that, if you need any aid, I will there to help you. Please feel free to contact me, either through email, or phone. I will definitely try to assist you. Good luck!*

1. Which is greater 17^14 or 31^11 ?
2. Let a, b, c be positive reals such that abc = 1. Prove that

a + b + c ≤ a²+ b²+ c².

1. (USA MO 1978)

a, b, c, d, e are real numbers such that

a + b + c + d + e = 8 &

a²+ b²+ c²+ d²+ e²= 16

What is the largest possible value of e?

1. Let P(x) be a polynomial with positive coefficients. Prove that if

P(⅟x)≥1/P(x)

holds for x=1, then it holds for all positive values of x.

1. Show that for all positive reals a, b, c, d,

⅟a +⅟b +4(⅟c) +16(⅟d) ≥64/ (a + b + c + d).

1. (USAMO 1980)

Show that for all non-negative reals a, b, c ≤ 1,

a/(b+c+1) + b/(c+a+1) +c/(a+b+1) + (1 - a)(1 - b)(1 - c) ≤ 1.

1. (IMO 1995)

a, b, c are positive reals with abc = 1. Prove that

1/(a³ (b + c)) + 1/(b³ (c + a)) + 1/(c³ (a + b)) ≥ 3/2.

1. (Poland 1995)

Let n be a positive integer. Compute the minimum value of the sum

x₁ + x₂²/2+ …. + xᵣʳ/r

where x₁, x₂,…, xᵣ are positive real numbers such that

⅟x₁ + ⅟x₂ + … + ⅟xᵣ = r.

1. (RMO 2001)

If x, y, z are sides of a triangle, then show that

|x² (y - z) + y² (z - x) + z² (x - y)| < xyz

1. (INMO 2001)

If a, b, c are positive real numbers such that abc = 1, prove that

a^(b + c).b^(c + a).c^(a + b) ≤ 1

1. (RMO 2002)

For any natural number n > 1, prove the inequality:

½ < 1/(n² + 1) + 2/(n² + 2) +3/(n² + 3) + … + n/(n² + n) < ½ + 1/(2n).

1. (INMO 2002)

Let x, y be positive real numbers such that x + y=2. Prove that

x³y³(x³ + y³) ≤ 2.

1. (RMO 2003)

Let a, b, c be three positive real numbers such that a + b + c = 1. Prove that among the three numbers (a – ab), (b – bc), (c – ca) there is one which is at most 1/4 and there is one which is at least 2/9.

1. (RMO 2004)

Let x, y be positive real numbers such that y + y³ ≤ x - x³. Prove that

1. y < x < 1
2. x² + y² < 1
3. (INMO 2005)

Let α and β be positive integers such that

43/197 < α/β < 17/77.

Find the minimum possible value of β.

1. (RMO 2005)

If a, b, c be three real numbers such that

|a - b| ≥ |c|,

|b – c| ≥ |a| and

|c – a| ≥ |b|,

then show that one of a, b, c is the sum of the other two.

1. (RMO 2006)

Let a, b, c be positive real numbers. Then show that,

(a² + 1)/ (b + c) + (b² + 1)/(c + a) + (c² + 1)/ (a + b) ≥ 3.

1. (RMO 2007)

Prove that:

1. 5 < 5^½ + 5^⅓ + 5^¼
2. 8 > 8^½ + 8^⅓ + 8^¼
3. n > n^½ + n^⅓ + n^¼ for all integers n ≥ 9.
4. (RMO 2011)

Find all pairs (x, y) of real numbers such that

16^(x + y²) + 16^(x² + y) = 1.

1. (INMO 2009)

Find all real numbers x such that

[x² + 2x] = [x]² + 2[x] .

(Here [x] denotes the largest integer not exceeding x.)

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**Note:** *Although I am providing the following inequalities as additional tools, it is my belief that you will hardly be using these inequalities. My Olympiad experience tells me that AM-GM, Cauchy-Schwarz, Jensen’s Inequalities will be more than sufficient to get you through.*

* **Rearrangement Inequality** :

Let a₁ ≤ a₂ ≤ ··· ≤ aᵣ and b₁ ≤ b₂ ≤ ··· ≤ bᵣ be two non decreasing sequences of real numbers. Then, for any permutation π of {1, 2, ..., n}, we have

S₁ = a₁b₁ + a₂b₂ + … + aᵣbᵣ is maximum, and

S₂ = a₁bᵣ + … + aᵣb₁ is the minimum.

* **Schur’s Inequality** :

Let a, b, c be nonnegative reals and r > 0. Then

aʳ(a − b)(a − c) + bʳ(b − c)(b − a) + cʳ(c − a)(c − b) ≥ 0

with equality if and only if a = b = c or some two of a, b, c are equal and the other is 0.

Remark - This can be improved considerably. However, they are not as well known as this form of Schur, and so should be proven whenever used on a contest.

* **Majorization Theorem** :

Let f : I → R be a convex function on I and suppose that the sequence

(x₁ , ... , xᵣ) majorizes the sequence (y₁ , ... , yᵣ), where x , y ∈ I. Then

f(x₁) + ··· + f(xᵣ) ≥ f(y₁) + ··· + f(yᵣ).

* **Popoviciu’s Inequality** :

Let f : I → R be convex on I, and let x, y, z ∈ I. Then for any positive reals p, q, r, then,

pf(x)+qf(y)+rf(z)+(p+q+r)f((px+qy+rz)/(p+q+r)) ≥

(p+q)f((px+qy)/(p+q))+(q+r)f((qy+rz)/(q+r))+(r+p)f(((rz+px)/(r+p))

* **Muirhead Inequalities** :

Suppose the sequence (a₁ , ... , aᵣ)majorizes the sequence (b₁ , ... , bᵣ).Then for any positive reals x₁ , ... , xᵣ ,

Σ(Πxᵃ) ≥ Σ(Πxᵇ)

where the sums are taken over all permutations of the r variables.

….and there are many more of such inequalities!

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