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Geometry.

Problems :

1. Let AC be a line segment in the plane and B a point between A and C. Construct isosceles triangles PAB and QBC on one side of the segment AC such that ∠APB = ∠BQC = 120◦ and an isosceles triangle RAC on the otherside of AC such that ∠ARC = 120◦. Show that PQR is an equilateral triangle.
2. Let ABC be a triangle in which AB = AC and ∠CAB = 90◦ . Suppose M and

N are points on the hypotenuse BC such that BM^2 + CN^2 = MN^2. Prove that ∠MAN = 45◦

3. The in-circle of triangle ABC touches the sides BC, CA and AB in K, L and M

respectively. The line through A and parallel to LK meets MK in P and the line

through A and parallel to MK meets LK in Q. Show that the line P Q bisects the

sides AB and AC of triangle ABC.

4 Let ABC be a triangle in which no angle is 90◦. For any point P in the plane

of the triangle, let A1, B1, C1 denote the reﬂections of P in the sides BC, CA, AB

respectively. Prove the following statements:

(a) If P is the incentre or an excentre of ABC, then P is the circumcentre ofA1B1C1;

(b) If P is the circumcentre of ABC, then P is the orthocentre of A1B1C1;

(c) If P is the orthocentre of ABC, then P is either the incentre or an excentre of

A1B1C1.

5 Let ABC be a triangle in which AB = AC and let I be its in-centre. Suppose

BC=AB +AI. Find ∠BAC.

1. . Let ABCDEF be a convex hexagon in which the diagonals AD, BE, CF are concurrent at O. Suppose the area of triangle OAF is the geometric mean of those of OAB and OEF; and the area of triangle OBC is the geometric mean of those of OAB and OCD. Prove that the area of triangle OED is the geometric mean of those of OCD and OEF.
2. Let ABC be a triangle with circum-circle Γ. Let M be a point in the interior of triangle ABC which is also on the bisector of ∠A. Let AM, BM, CM meet Γ in A1, B1, C1 respectively. Suppose P is the point of intersection of A1C1 with AB; and Q is the point of intersection of A1B1 with AC. Prove that P Q is parallel to BC..
3. Let ABC be an acute-angled triangle with altitude AK. Let H be its ortho-centre and O be its circum-centre. Suppose KOH is an acute-angled triangle and P its circum-Centre Let Q be the reﬂection of P in the line HO. Show that Q lies on the line joining the mid-points of AB and AC.

Circles [Q_1](http://www.artofproblemsolving.com/Forum/code.php?hash=facf17f6b7804da9278de91d4a7370f0c9153634&sid=1f0e151601b3f6c8e756cbd37f392359) and [Q_2](http://www.artofproblemsolving.com/Forum/code.php?hash=2e1f27afc9f839fd3d8a3215c6925a2c51a1b45e&sid=1f0e151601b3f6c8e756cbd37f392359) are tangent to each other externally at [T](http://www.artofproblemsolving.com/Forum/code.php?hash=c2c53d66948214258a26ca9ca845d7ac0c17f8e7&sid=1f0e151601b3f6c8e756cbd37f392359). Points [A](http://www.artofproblemsolving.com/Forum/code.php?hash=6dcd4ce23d88e2ee9568ba546c007c63d9131c1b&sid=1f0e151601b3f6c8e756cbd37f392359) and [E](http://www.artofproblemsolving.com/Forum/code.php?hash=e0184adedf913b076626646d3f52c3b49c39ad6d&sid=1f0e151601b3f6c8e756cbd37f392359) are on [Q_1](http://www.artofproblemsolving.com/Forum/code.php?hash=facf17f6b7804da9278de91d4a7370f0c9153634&sid=1f0e151601b3f6c8e756cbd37f392359), lines [AB](http://www.artofproblemsolving.com/Forum/code.php?hash=06d945942aa26a61be18c3e22bf19bbca8dd2b5d&sid=1f0e151601b3f6c8e756cbd37f392359) and [DE](http://www.artofproblemsolving.com/Forum/code.php?hash=ce3e4bed2954adbf05f8edfaf1a4c0cc0cea70e9&sid=1f0e151601b3f6c8e756cbd37f392359) are tangent to [Q_2](http://www.artofproblemsolving.com/Forum/code.php?hash=2e1f27afc9f839fd3d8a3215c6925a2c51a1b45e&sid=1f0e151601b3f6c8e756cbd37f392359)at [B](http://www.artofproblemsolving.com/Forum/code.php?hash=ae4f281df5a5d0ff3cad6371f76d5c29b6d953ec&sid=1f0e151601b3f6c8e756cbd37f392359) and [D](http://www.artofproblemsolving.com/Forum/code.php?hash=50c9e8d5fc98727b4bbc93cf5d64a68db647f04f&sid=1f0e151601b3f6c8e756cbd37f392359), respectively, lines [AE](http://www.artofproblemsolving.com/Forum/code.php?hash=c1e94b2615a2d560afd200d391867a6340c9c871&sid=1f0e151601b3f6c8e756cbd37f392359) and [BD](http://www.artofproblemsolving.com/Forum/code.php?hash=0f4d56d1e20778bf2e1052ecb3219509238fb660&sid=1f0e151601b3f6c8e756cbd37f392359) meet at point [P](http://www.artofproblemsolving.com/Forum/code.php?hash=511993d3c99719e38a6779073019dacd7178ddb9&sid=1f0e151601b3f6c8e756cbd37f392359). Prove that   
(1) [\frac{AB}{AT}=\frac{ED}{ET}](http://www.artofproblemsolving.com/Forum/code.php?hash=fd966cfa2e613a18a7829e5cdc8a00286bfa4ba3&sid=1f0e151601b3f6c8e756cbd37f392359);   
(2) [\angle ATP + \angle ETP = 180^{\circ}](http://www.artofproblemsolving.com/Forum/code.php?hash=7c1941a58633e9dbaef2586ef662046ce491ba5c&sid=1f0e151601b3f6c8e756cbd37f392359).

1. If [x](http://www.artofproblemsolving.com/Forum/code.php?hash=11f6ad8ec52a2984abaafd7c3b516503785c2072&sid=1f0e151601b3f6c8e756cbd37f392359), [y](http://www.artofproblemsolving.com/Forum/code.php?hash=95cb0bfd2977c761298d9624e4b4d4c72a39974a&sid=1f0e151601b3f6c8e756cbd37f392359), [z](http://www.artofproblemsolving.com/Forum/code.php?hash=395df8f7c51f007019cb30201c49e884b46b92fa&sid=1f0e151601b3f6c8e756cbd37f392359) are positive real numbers, prove that   
   [\left(x + y + z\right)^2 \left(yz + zx + xy\right)^2 \leq 3\left(y^2 + yz + z^2\right)\left(z^2 + zx + x^2\right)\left(x^2 + ...](http://www.artofproblemsolving.com/Forum/code.php?hash=29129a4cff3451f6c956d4be4f706f585d53fafe&type=1&sid=1f0e151601b3f6c8e756cbd37f392359)(Use geometric interpretation ! )
2. In an acute angled triangle [ABC](http://www.artofproblemsolving.com/Forum/code.php?hash=3c01bdbb26f358bab27f267924aa2c9a03fcfdb8&sid=1f0e151601b3f6c8e756cbd37f392359), [\angle A = 30^{\circ}](http://www.artofproblemsolving.com/Forum/code.php?hash=28f710d5f4080167ca7e1c0e622a13c5dac4194d&sid=1f0e151601b3f6c8e756cbd37f392359), [H](http://www.artofproblemsolving.com/Forum/code.php?hash=7cf184f4c67ad58283ecb19349720b0cae756829&sid=1f0e151601b3f6c8e756cbd37f392359) is the orthocenter, and [M](http://www.artofproblemsolving.com/Forum/code.php?hash=c63ae6dd4fc9f9dda66970e827d13f7c73fe841c&sid=1f0e151601b3f6c8e756cbd37f392359) is the midpoint of [BC](http://www.artofproblemsolving.com/Forum/code.php?hash=a7a051355c3e05fb5fb80de30930bee9d2eb7847&sid=1f0e151601b3f6c8e756cbd37f392359). On the line [HM](http://www.artofproblemsolving.com/Forum/code.php?hash=0bdcc8f4a8b02f74d69a9ecf143cf15f28f98cfc&sid=1f0e151601b3f6c8e756cbd37f392359), take a point [T](http://www.artofproblemsolving.com/Forum/code.php?hash=c2c53d66948214258a26ca9ca845d7ac0c17f8e7&sid=1f0e151601b3f6c8e756cbd37f392359) such that [HM = MT](http://www.artofproblemsolving.com/Forum/code.php?hash=22709c646c6fed2ea78fab2da286333d76c9f047&sid=1f0e151601b3f6c8e756cbd37f392359). Show that [AT = 2 BC](http://www.artofproblemsolving.com/Forum/code.php?hash=1d5ba6ff180318ab6217fe9abd8453c199c728b9&sid=1f0e151601b3f6c8e756cbd37f392359).
3. The diagonals [AC](http://www.artofproblemsolving.com/Forum/code.php?hash=b1fb3bec6fdb22e19a94fe4c6c4481ccba2ee9f0&sid=1f0e151601b3f6c8e756cbd37f392359) and [BD](http://www.artofproblemsolving.com/Forum/code.php?hash=0f4d56d1e20778bf2e1052ecb3219509238fb660&sid=1f0e151601b3f6c8e756cbd37f392359) of a cyclic quadrilateral [ABCD](http://www.artofproblemsolving.com/Forum/code.php?hash=fb2f85c88567f3c8ce9b799c7c54642d0c7b41f6&sid=1f0e151601b3f6c8e756cbd37f392359) intersect at [P](http://www.artofproblemsolving.com/Forum/code.php?hash=511993d3c99719e38a6779073019dacd7178ddb9&sid=1f0e151601b3f6c8e756cbd37f392359). Let [O](http://www.artofproblemsolving.com/Forum/code.php?hash=08a914cde05039694ef0194d9ee79ff9a79dde33&sid=1f0e151601b3f6c8e756cbd37f392359) be the circumcenter of triangle [APB](http://www.artofproblemsolving.com/Forum/code.php?hash=2fbce66c8fbf34e8aee5f3fb9774360518dc54c5&sid=1f0e151601b3f6c8e756cbd37f392359) and [H](http://www.artofproblemsolving.com/Forum/code.php?hash=7cf184f4c67ad58283ecb19349720b0cae756829&sid=1f0e151601b3f6c8e756cbd37f392359) be the orthocenter of triangle [CPD](http://www.artofproblemsolving.com/Forum/code.php?hash=4f3cee106e92f4d69824c96ef74d7f1561fda110&sid=1f0e151601b3f6c8e756cbd37f392359). Show that the points [H,P,O](http://www.artofproblemsolving.com/Forum/code.php?hash=9b0896ffcd444ff38ab4f8c1477a89eec2bfeadb&sid=1f0e151601b3f6c8e756cbd37f392359) are collinear.
4. Let [ABCDE](http://www.artofproblemsolving.com/Forum/code.php?hash=7be07aaf460d593a323d0db33da05b64bfdcb3a5&sid=1f0e151601b3f6c8e756cbd37f392359) be a convex pentagon such that [BC \parallel AE,](http://www.artofproblemsolving.com/Forum/code.php?hash=59cce236d141aed1bc69421cf2eddc7b576684d7&sid=1f0e151601b3f6c8e756cbd37f392359) [AB = BC +  AE,](http://www.artofproblemsolving.com/Forum/code.php?hash=374452405eee576ee91e01354245adb5f3abb309&sid=1f0e151601b3f6c8e756cbd37f392359) and [\angle ABC = \angle CDE.](http://www.artofproblemsolving.com/Forum/code.php?hash=07dc8cea1e097fe13e9bd5c4d24b5f38cdd5249f&sid=1f0e151601b3f6c8e756cbd37f392359) Let [M](http://www.artofproblemsolving.com/Forum/code.php?hash=c63ae6dd4fc9f9dda66970e827d13f7c73fe841c&sid=1f0e151601b3f6c8e756cbd37f392359) be the midpoint of [CE,](http://www.artofproblemsolving.com/Forum/code.php?hash=5b39639fdd3cec43ad46514a1e3b80cc1cc2becb&sid=1f0e151601b3f6c8e756cbd37f392359) and let [O](http://www.artofproblemsolving.com/Forum/code.php?hash=08a914cde05039694ef0194d9ee79ff9a79dde33&sid=1f0e151601b3f6c8e756cbd37f392359) be the circumcenter of triangle [BCD.](http://www.artofproblemsolving.com/Forum/code.php?hash=ad1c9125b7634a5f644de37a3723845ccf931dd0&sid=1f0e151601b3f6c8e756cbd37f392359) Given that [\angle DMO = 90^{\circ},](http://www.artofproblemsolving.com/Forum/code.php?hash=0d80456c3d969fe9a41bcc894440c541e8d40b7d&sid=1f0e151601b3f6c8e756cbd37f392359) prove that [2 \angle BDA = \angle CDE.](http://www.artofproblemsolving.com/Forum/code.php?hash=c39adbce6c83cfef576f12b10da89a5340caf1aa&sid=1f0e151601b3f6c8e756cbd37f392359)
5. Let [AL](http://www.artofproblemsolving.com/Forum/code.php?hash=cf681cfbf3c282528fea28596536f592630842e7&sid=1f0e151601b3f6c8e756cbd37f392359) and [BK](http://www.artofproblemsolving.com/Forum/code.php?hash=5b3e3191fd7659c5e63ab6620a903048b48bcb56&sid=1f0e151601b3f6c8e756cbd37f392359) be the angle bisectors in a non-isosceles triangle [ABC,](http://www.artofproblemsolving.com/Forum/code.php?hash=35dbea4ed15b3e87e5322663a6fdb8a179de527e&sid=1f0e151601b3f6c8e756cbd37f392359) where [L](http://www.artofproblemsolving.com/Forum/code.php?hash=d160e0986aca4714714a16f29ec605af90be704d&sid=1f0e151601b3f6c8e756cbd37f392359) lies on [BC](http://www.artofproblemsolving.com/Forum/code.php?hash=a7a051355c3e05fb5fb80de30930bee9d2eb7847&sid=1f0e151601b3f6c8e756cbd37f392359) and [K](http://www.artofproblemsolving.com/Forum/code.php?hash=a7ee38bb7be4fc44198cb2685d9601dcf2b9f569&sid=1f0e151601b3f6c8e756cbd37f392359) lies on [AC.](http://www.artofproblemsolving.com/Forum/code.php?hash=4bf2c9ca8c8dc75a97464a39d8af7405583d94c3&sid=1f0e151601b3f6c8e756cbd37f392359) The perpendicular bisector of [BK](http://www.artofproblemsolving.com/Forum/code.php?hash=5b3e3191fd7659c5e63ab6620a903048b48bcb56&sid=1f0e151601b3f6c8e756cbd37f392359) intersects the line [AL](http://www.artofproblemsolving.com/Forum/code.php?hash=cf681cfbf3c282528fea28596536f592630842e7&sid=1f0e151601b3f6c8e756cbd37f392359) at [M](http://www.artofproblemsolving.com/Forum/code.php?hash=c63ae6dd4fc9f9dda66970e827d13f7c73fe841c&sid=1f0e151601b3f6c8e756cbd37f392359). Point [N](http://www.artofproblemsolving.com/Forum/code.php?hash=b51a60734da64be0e618bacbea2865a8a7dcd669&sid=1f0e151601b3f6c8e756cbd37f392359) lies on the line [BK](http://www.artofproblemsolving.com/Forum/code.php?hash=5b3e3191fd7659c5e63ab6620a903048b48bcb56&sid=1f0e151601b3f6c8e756cbd37f392359) such that [LN](http://www.artofproblemsolving.com/Forum/code.php?hash=530dea69f55c0fe61ae6e24c501e8cb4e66967f7&sid=1f0e151601b3f6c8e756cbd37f392359) is parallel to [MK.](http://www.artofproblemsolving.com/Forum/code.php?hash=bc57319a77faa8c8950ea7cdec87a1e214a4c6cb&sid=1f0e151601b3f6c8e756cbd37f392359)Prove that [LN=NA.](http://www.artofproblemsolving.com/Forum/code.php?hash=0005d648ef84fc5b3570859d4514e25d73514174&sid=1f0e151601b3f6c8e756cbd37f392359)
6. Show that [3^{2008} + 4^{2009}](http://www.artofproblemsolving.com/Forum/code.php?hash=59b0354511e0ef4d2384532c032014ffd958cdcb&sid=1f0e151601b3f6c8e756cbd37f392359) can be written as product of two positive integers each of which is larger than [2009^{182}](http://www.artofproblemsolving.com/Forum/code.php?hash=90f072340e459073f7a7f75fd6c4943cd24d556e&sid=1f0e151601b3f6c8e756cbd37f392359).

(Not a geometry problem ! Use Sophie-Germain Identity of number theory ).

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Few Interesting Geometry Theorems.

The lines from the points of tangency of the incircle to the vertices of a triangle are concurrent.( Gergonne Point)

If three points are chosen, one on each side of a triangle then the three circles determined by a vertex and the two points on the adjacent sides meet at a point.(Miquel Theorem)

The adjacent trisectors of the angles of a triangle are concurrent by pairs at the vertices of an equilateral triangle.(Morley’s theorem)

The isogonal conjugates of a set of cevians are also concurrent.(Concurrency of isogonal conjugates).

Two triangles are centrally perspective if and only if they are axially perspective.(Desargues Theorem)

Let A, B, C, D, E, F be points on a circle (which are not necessarily in cyclic order). Let P=AB∩DE, Q=BC∩EF, R=CD∩FA. Then P,Q,R are collinear.(Pascal’s Theorem of cyclic hexagon)

 Given one set of collinear points *A*, *B*, *C*, and another set of collinear points *a*, *b*, *c*, then the intersection points *X*, *Y*, *Z* of line pairs *Ab* and *aB*, *Ac* and *aC*, *Bc* and *bC* are collinear.( Pappus theorem)

For a cyclic quadrilateral the maltitudes intersect in a single point, called the anticenter.(Anticenter of quadrilateral)

>Perpendiculars from mid-point of one diagonal to other intersect at the anti center.

> Construct 9-point circles of the four triangles ∆ABD, ∆BCD, ∆ABC, and ∆ADC.The 4 circles intersect at the anticenter.

> The centers of the 9-point circles are concyclic with anticenter.

The nine-point circle of any triangle is tangent internally to the incircle and tangent externally to the three excircles(Feuerbach’s Theorem).