Combinatorics

Biswajit Paria

Problems

1. Find the number of integer solutions to the equation

$$x_1 + x_2 + \cdots + x_n = k \tag{1}$$

such that the following properties hold.

$$x_i \geq i - 1 \quad \forall i = 1, 2, \ldots, n. \tag{2}$$

2. Find the number of positive integer solutions to the equation

$$(x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n) = p, \tag{3}$$

where $n \in \mathbb{N}$ and p is a prime.

3. Vandermonde's Identity

$$\sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i} = \binom{m+n}{r} \tag{4}$$

4. Prove the following identities

   (a) $\displaystyle\sum_{i=1}^{n} i\binom{n}{i} = n \cdot 2^{n-1}$

   (b) $\displaystyle\sum_{i=1}^{n} i^2\binom{n}{i} = n(n+1) \cdot 2^{n-2}$

5. (IMO 1981) Let $1 \leq r \leq n$ and consider all r-element subsets of $\{1, 2, \ldots, n\}$. Each of these subsets has a smallest number. Let $F(n, r)$ denote the arithmetic mean of these numbers. Prove that

$$F(n, r) = \frac{n+1}{r+1} \tag{5}$$

6. A monotonic path is defined as a sequence of *Up* and *Down* steps only. The number of monotonic paths from one corner to the opposite corner of an $n \times n$ grid which lie on or below the main diagonal is denoted by the nth Catalan number C_n. Show that

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{6}$$

7. (AHSME 1991) If A, B, C are sets for which

$$n(A) + n(B) + n(C) = n(A \cup B \cup C) \text{ and } |A| = |B| = 100 \tag{7}$$

then what is the minimum value of $|A \cap B \cap C|$?

8. A basketball team consists of 12 pairs of twin brothers. In how many ways can all 24 players stand in a circle such that all pairs of twin brothers are neighbours?

9. Let n be an integer with $n \geq 3$. Let $P_1P_2 \ldots P_n$ be a regular n-sided polygon inscribed in a circle ω. Three points P_i, P_j and P_k are randomly chosen, where i, j, k are distinct integers between 1 and n, inclusive. What is the probability that the $\triangle P_iP_jP_k$ is obtuse.

10. Let $A_1A_2 \ldots A_12$ be the vertices of a regular dodecagon. How many distinct squares in the plane have at least two vertices in the set $\{A_1, A_2, \ldots, A_12\}$?

11. A triangular grid is obtained by tiling an equilateral triangle of side length n by n^2 equiateral triangles of side length 1. Determine the number of parallelograms bounded by the line segments of the grid.

12. There are n envelopes each with an letter assigned to it. Let D_n denote the number of ways in which the letters can be reinserted in the envelopes such that no envelope contains the letter assigned to it. Show that

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right) \tag{8}$$

13. We draw all diagonals of a convex n-gon. Suppose no three diagonals pass through a point. Find the number of regions into which the polygon is divided.

14. Find the smallest positive integer k such that if k squares of a $n \times n$ chessboard are colored, then there will exist three colored squares which form a right angled triangle with sides parallel to the edges of the chessboard.

15. There are $6n$ points on a line segment, of which any $3n$ points are colored Blue and the other $3n$ points are colored Red. Show that there exist $2n$ consecutive points with n points colored blue and n points colored red.

16. There are 1982 persons in a party. Among any group of four persons there is a person who knows the other three. What is the minimum number of people who know everyone else.

17. The 20 members of a local tennis club have scheduled exactly 14 two person games among themselves, with each member playing atleast one game. Prove that within this schedule there must be a set of 6 games with 12 distinct players.

18. Sixty students took AIME at PEA in 2003. The possible scores on the AIME are integers between 0 t 15 inclusive. Let $a_1, a_2, \ldots, a_{60}$ denote the students score on the AIME. For $k = 0, 1, \ldots, 15$, let b_k denote the number of students with a score of at least k. Show that

$$a_1 + a_2 + \cdots + a_{60} = b_1 + b_2 + \cdots + b_{15} \tag{9}$$

19. There are n points in a plane such that no three of them are collinear. Show that there are atleast $\binom{n-3}{2}$ convex quadrilaterals with the vertices at these points.

20. Prove that among any 16 distinct positive integers not exceeding 100 there are distinct positive integers a, b, c, d, such that $a + b = c + d$.

21. For any positive integer n. Prove that

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\binom{n}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \tag{10}$$

22. Let F_n denote the nth Fibonacci number defined by $F_0 = F_1 = 1$, $F_n = F_{n-1} + F_{n-1} \quad (n \geq 2)$. Prove that for $n \geq 0$

$$\sum_{k=0}^{n} \binom{n-k+1}{k} = F_{n+1} \tag{11}$$

23. (USA-TST 2000, Kvant) Let n be a positive integer. Prove that

$$\sum_{i=0}^{n} \binom{n}{i}^{-1} = \frac{n+1}{2^{n+1}} \sum_{i=1}^{n+1} \frac{2^i}{i} \tag{12}$$

24. Show that

$$\binom{n+1}{k}^{-1} + \binom{n+1}{k+1}^{-1} = \frac{n+2}{n+1}\binom{n}{k}^{-1} \tag{13}$$

25. (Lucas Theorem) Let p be a prime, and let n be a positive integer with $n = n_0 + n_1 p + \cdots + n_m p^m$, where $0 \leq n_0, n_1, \ldots, n_m < p$. Also, write $i = i_0 + i_1 p + \cdots + i_m p^m$, where $0 \leq i_0, i_1, \ldots, i_m < p$. Then

$$\binom{n}{i} \equiv \prod_{j=0}^{m} \binom{n_j}{i_j} \pmod{p} \tag{14}$$

26. Let
$$\prod_{n=1}^{1996}\left(1+nx^{3^n}\right)=1+a_1x^{k_1}+a_2x^{k_2}+\cdots+a_mx^{k_m}, \tag{15}$$
where $a_1,a_2,\ldots,a_m$ are non-zero and $k_1<k_2<\cdots<k_m$. Find a_{1234}.

27. (A straightforward generalization of P. 26) Let b be a positive integer such that $b>1$. Let
$$\prod_{n\geq 1}\left(1+nx^{b^n}\right)=1+a_1x^{k_1}+a_2x^{k_2}+\cdots+a_ix^{k_i}+\cdots, \tag{16}$$
where $a_i>0$ and $k_i<k_{i+1}$ for $i>0$. Let $t=t_1+t_2\cdot 2^1+\cdots+t_m\cdot 2^{m-1}$, where $0\leq t_1,t_2,\ldots,t_m<2$. Let p_i be a positive integer such that for $i=1,2,\ldots,m$, $p_i=\begin{cases} i & \text{if } t_i>0 \\ 1 & \text{otherwise.}\end{cases}$.

Show that
$$a_t=\prod_{i=1}^{m}p_i. \tag{17}$$

28. (USAMO 1996) An ordered n-tuple $(x_1,x_2,\ldots,x_n)$ in which each term is either 0 or 1 is called a *binary sequence of length* n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to $0,1,0$ in that order. Let b_n be the number of binary sequences of length n which have no four consecutive terms equal to $0,0,1,1$ or $1,1,0,0$ in that order. Prove that $b_{n+1}=2a_n$.

29. Let k and n be positive integers, and let $S=1,2,\ldots,n$. A subset of S is called *skipping* if it does not contain consecutive integers. How many k-element skipping subsets are there altogether?

30. (Tower of Hanoi) We are given a small board into which three rods have been inserted and a set of n disks of different diameters with holes so that they can fit over the rods. Initially all disks are on the same rod, with the largest disk on the bottom, the second largest disk above the largest disk, the third largest disk above the second largest disk and so on. Adrian is asked to move the tower to one of the other rods in such a way that during the entire process no disk is above a smaller disk on the same rod. What is the minimum number of moves that Adrian has to make?[1]

31. Each unit square of a $2\times n$ unit square grid is to be colored blue or red such that no 2×2 red square is obtained. Let c_n denote the number of different colorings. Determine with proof, the greatest integer k such that $3^k|c_{2001}$.

32. Let S be a finite set of n elements, and let k be a positive integer. Determine the number of ordered k-tuples $(S_1,S_2,\ldots,S_k)$ of subsets of S such that $\cap_{i=1}^{k}S_i=\phi$.

33. (Euler's Phi function) Let $\phi(n)$ denote the number of positive integers less than n and relatively prime to n. Show that
$$\phi(n)=n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right) \tag{18}$$
where
$$n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}\quad ,\alpha_i>0\ \forall\ 1\leq i\leq k \tag{19}$$

34. A 15×15 square is tiled with unit squares. Each vertex is colored either blue of red. There are 133 red points. Two of those red points are corners of the original square, and another 32 red points are on the sides. The sides of the unit squares are colored according to the following rule: If both endpoints are red, then it is colored red; if the points are both blue, then it is colored blue; if one point is red and the other is blue, then it is colored yellow. Suppose that there are 196 yellow sides. How many blue segments are there?

35. Let X be a finite set with $|X|=n$, and let $A_1,A_2,\ldots,A_m$ be three-element subsets of X such that $|A_i\cap A_j|\leq 1$ for all $i\neq j$. Show that there exists a subset A of X with at least $\lfloor\sqrt{2n}\rfloor$ elements containing none of the A_i.

36. (Sperner) Let S be a set with $|S|=n$. Assume that $S_1,S_2,\ldots,S_m$ are subsets of S such that $S_i\not\subseteq S_j$ for $i\neq j$. Then
$$m\leq\binom{n}{\lfloor\frac{n}{2}\rfloor} \tag{20}$$

[1] Search for 'Tower of Hanoi' on the internet for a picture of it.