RMO Workshop : Algebra

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1. Prove that $2^{3^n} + 1$ is divisible by 3^{n+1} for all integers $n \geq 0$.

2. Prove that $f(n) = g(n)$ For all $n \in \mathbb{N}$ where
   $f(n) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n}$ and $g(n) = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n-1} + \frac{1}{2n}$.

3. Let n = 2^k. Prove that : we can select n numbers from a set of any (2n - 1) numbers, such that the sum of the selected numbers is divisible by n.

4. Let α, β be roots of the equation $y^2 - my + 1 = 0$ where m is and odd integer.
   Let $\lambda_n = \alpha^n + \beta^n \ \forall n \geq 0$. Prove that

   (a) λ_n is an integer.

   (b) $gcd(\lambda_n, \lambda_{n+1}) = 1 \ \forall n \geq 0$

5. If r is an approximation of $\sqrt{5}$. Prove that $\frac{2r+5}{r+2}$ is a better approximation.
   Generalize it for $\sqrt{a}$ where $a \in \mathbb{N}$.

6. Find S_n where $S_n = \frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6} + \ldots + \frac{1}{n(n+1)(n+2)(n+3)}$

7. Prove that :
   $$\sum_{k=0}^{n} \binom{n+k}{k} \times \frac{1}{2^{-k}} = 2^n$$

8. If $x^3 + px^2 + qx + 1 = 0$ has 3 real zeros then prove that $p^2 \geq 3q$.

9. Let f(x) be a polynomial with integer coefficients. Prove that it has no integral zeros if f(0) and f(1) are both odd.

10. Let p(x) $= x^n + a_1x^{n-1} + \ldots + a_{n-1}x + 1$ and $\forall i,\ a_i \geq 0$ has n real roots.
    Prove that $P(2) \geq 3^n$

11. Prove that $1 < \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n} + \frac{1}{3n+1} < 2$

12. Let P(x) be a polynomial of degree n so that $P(k) = \frac{k}{k+1} \ \forall k = 0, 1, \ldots, n$.
    Find P(n+1).

13. Let a, b, c be distinct integers. P is an integral polynomial. Show that : $P(a) = b,\ P(b) = c,\ P(c) = a$ cannot all be true.

14. Prove that P(x) $= 1 + \frac{x}{1} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$ has no multiple roots.

15. Does there exist a sequence of positive reals $\{a_n\}$ such that both $\sum a_n$ and $\sum \frac{1}{n^2 a_n}$ are convergent.