School of Mathematical and Computational Sciences
Indian Association for the Cultivation of Science

Master's/Integrated Master's-PhD Program/ Integrated Bachelor's-Master's Program/PhD Course

Theory of Computation II: COM 5108

Tutorial VIII (02 November 2023)

Instructor: Goutam Biswas *Autumn Semester 2023*

1. According to the *time hierarchy theorem*, for any time constructible $f : \mathbb{N}_0 \to \mathbb{N}_0$, there is a language $A \in DTIME(f(n))$ but not decidable in time $o(f(n)/log f(n))$.
   (a) Are $2^n, 2^{n+1}, 2^{2n}$ time constructible?
   (b) Is $DTIME(2^n)$ a proper subset of $DTIME(2^{2n})$?
   (c) Is $DTIME(2^n)$ a proper subset of $DTIME(2^{n+1})$?

2. Let $A_{TM} = \{< M, x >: \text{ the TM } M \text{ accepts } x\}$ and $A_{\neg\emptyset} = \{< M >: L(M) \neq \emptyset\}$. Show that $A_{\neg\emptyset} \leq_m A_{TM}$.

3. Prove that
   (a) If $A \leq_m B$, then $A \leq_T B$.
   (b) If $A \leq_m^p B$, then $A \leq_T^p B$.

4. Prove that $SAT_H = \{\phi 0 1^{n^{H(n)}} : \phi \in SAT, |\phi| = n\} \in \mathbf{NP}$. The function $H(n)$ is time constructible.

5. The language $SAT_H = \{\phi 0 1^{n^{H(n)}} : \phi \in SAT, |\phi| = n\}$ is **NP**-complete if $H(n)$ is bounded i.e. $f(n) = n^{H(n)}$ is a polynomial of n. But it is in **P** if $f(n)$ is super-polynomial.
   How do you design $H(n)$ so that under the assumption $\mathbf{P} \neq \mathbf{NP}$, there will be a contradiction, if $SAT_H \in \mathbf{P}$ or SAT_H is **NP**-complete.

6. The function $H(n)$ is defined as follows:

$$H(n) = \begin{cases} i, & i = \min\{j : j \text{ satisfies } C\} \\ \log\log n, & \text{otherwise.} \end{cases}$$

   C: j is a natural number, $1 \leq j < \log\log n$, such that the TM M_j decides the membership of all $x \in \{0,1\}^*$, $|x| \leq \log n$ in SAT_H, within $j \times |x|^j$ steps.

   Base case may be suitably defined.

   Prove the following:
   (a) If $SAT_H \in \mathbf{P}$, $H(n)$ is bounded.
   (b) If $H(n) \leq c$, then $SAT_H \in \mathbf{P}$.