School of Mathematical and Computational Sciences

Indian Association for the Cultivation of Science

Master's/Integrated Master's-PhD Program/ Integrated Bachelor's-Master's Program/PhD Course

Theory of Computation II: COM 5108

Tutorial VI (21 September 2023)

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1. 5-SUBSET-SSUM $= \{< S, t >: \ S = \{x_1, \cdots, x_n : x_i \in \mathbb{N}\}, \ \exists S' = \{y_1, \cdots, y_5\} \subseteq S, \ \sum_{i=1}^{5} y_i = t\}$. Is 5-SUBSET-SSUM **NP**-complete?

   Ans. 5-SUBSET-SSUM is in **P** as as there are $\binom{n}{5}$ such subsets and sum of their elements can be tested in polynomial time. But proving or disproving 5-SUBSET-SSUM **NP**-complete is a problem, as the answer to $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ is unknown. If 5-SUBSET-SSUM is **NP**-complete, then all **NP** problems can be reduced to it. And because 5-SUBSET-SSUM $\in \mathbf{P}$, all of them can be solved in polynomial time i.e. $\mathbf{NP} \subseteq \mathbf{P} \Rightarrow \mathbf{NP} = \mathbf{P}$.
   We also know that if $\mathbf{P} = \mathbf{NP}$ and a language L is such that $L \in \mathbf{P}$, $L \neq \phi$, $L \neq \Sigma^*$, then L is NP- complete. The contrapositive statement is, if such L is not **NP**-complete, then $\mathbf{P} \neq \mathbf{NP}$. 5-SUBSET-SSUM is such a language.
   So the answer to "Is 5-SUBSET-SSUM **NP**-complete?" is unknown.

2. PATH $= \{< G, s, d >: G$ is a graph where there is a path from s to $d\}$. PATH is known to be in **P**. What is the repercussion if it is proved that PATH is not **NP**-complete.

   Ans. The answer is similar to (1). We know that if $\mathbf{P} = \mathbf{NP}$ and a language $L \neq \phi$ or Σ^*, then L is **NP**-complete.
   PATH satisfies the requirements of L. So a proof of PATH $\notin$ **NP**-complete implies that $\mathbf{N} \neq \mathbf{NP}$.

3. Solve the following problems.

   (i) What is the value of $n^{\left(\frac{1}{\ln n}\right)}$ $n > 1$?

   (ii) Show that $n^{\left(\frac{1}{c}\right)^{\log \log n}}$ is bounded above (less than a constant), where $c \geq 3$ is a constant and $n \geq e$.

   Ans.

   (i) Let
   $$n^{1/\ln n} = x \Rightarrow \ln x = \frac{1}{\ln n} \cdot \ln n = 1 \Rightarrow x = e.$$

   (ii) $c^{\ln \ln n} = \left(e^{\ln c}\right)^{\ln \ln n} = (e^{\ln \ln n})^{\ln c} = (\ln n)^d$, where $d = \ln c > 1$.
   Therefore
   $$n^{\frac{1}{c^{\ln \ln n}}} = n^{\frac{1}{(\ln n)^d}} < n^{\left(\frac{1}{\ln n}\right)} < e.$$

4. Prove that if there is a polynomial time reduction from SAT to SAT such that $\phi \mapsto \psi$ and $|\psi| = \sqrt[3]{|\phi|}$, then SAT is in $\mathbf{P}$.

   Ans. Let the reduction be bounded by n^c steps. If we perform the reduction $\log\log n$ times. The number of steps are $O(n^{c+1})$. The reduced formula is less than a constant length. The satisfiability of a formula of constant length can be checked by a constant size table look-up in $O(1)$ time. So the membership of ϕ can be tested in polynomial time through the reduction.

5. The language of undirected Hamiltonian path (UHAMPATH) is defined as follows.

   $UHAMPATH = \{< G, s, d >$: G is an undirected graph and there is a Hamiltonian path from s to $d\}$.

   (i) Show that UHAMPATH $\in$ **NP**.

   (ii) Show that UHAMPATH is **NP**- hard.

   Ans.

   (i) The certificate is simply sequence of nodes starting with s and ending at d. It can be verified in polynomial time that the sequence forms a Hamiltonian path.

   (ii) HAMPATH is reduced to UHAMPATH as follows. Let G be a *directed graph* and s and d are two of its nodes. We map $< G, s, d >$ to $< G', s', d' >$ where G' is an undirected graph.
   For each $u \in V(G) \setminus \{s, d\}$, there are three nodes u_i, u_m, u_o in $V(G')$. For $s, d \in V(G)$ we have $s' = s_o, d' = d_i \in V(G')$.
   Edges $\{u_i, u_m\}, \{u_m, u_o\} \in E(G')$ for all $u \in V(G)$. If $(u, v) \in E(G)$, then $\{u_o, v_i\} \in E(G')$. So

   $$V(G') = \{s_o, d_i\} \cup \{u_i, u_m, u_o : \ u \in V(G) \setminus \{s, d\}\},$$

   $$E(G') = \{\{u_i, u_m\}, \{u_m, u_o\} : \ u \in V(G)\} \cup \{\{u_o, v_i\} : \ (u, v) \in E(G)\}.$$

   $s, u_1, u_2, \cdots, u_k, d$ is a Hamiltonian path in G if and only if $s_o, u_{1i}, u_{1m}, u_{1o}, u_{2i}, u_{2m}, u_{2o}, \cdots, u_{ki}, u_{km}, u_{ko}, d_i$ is a Hamiltonian path in G'.