School of Mathematical and Computational Sciences

Indian Association for the Cultivation of Science

Master's/Integrated Master's-PhD Program/ Integrated Bachelor's-Master's Program/PhD Course

Theory of Computation II: COM 5108

Tutorial III (24 August 2023)

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1. Use *diagonalization*(directly) to show that there is no bijection from $\mathbb{N} \to \mathscr{P}\mathbb{N}$.

   Ans. Suppose there is a bijection $f : \mathbb{N} \to \mathscr{P}\mathbb{N}$ i.e. f is one-one and onto, no two positive integers are mapped to the same subset of $\mathbb{N}$ and for each $S \subseteq \mathbb{N}$, there is an $m \in \mathbb{N}$ such that $f(m) = S$. The following infinite table shows the mapping $n \mapsto f(n) = S_n$ through its characteristic function.

n	$m \in f(n)$									
	1	2	3	4	5	6	7	8	9	$\cdots$
1	0	1	1	0	1	0	0	1	1	$\cdots$
2	0	0	1	0	1	1	0	0	1	$\cdots$
3	0	0	1	0	1	1	0	0	1	$\cdots$
4	1	1	1	0	1	0	0	0	0	$\cdots$
5	**0**	**0**	**1**	**1**	**0**	**1**	**0**	**1**	**1**	$\cdots$
6	0	1	0	0	1	1	1	0	1	$\cdots$
7	1	1	1	0	0	0	1	0	0	$\cdots$
8	1	0	0	1	1	1	1	0	0	$\cdots$
9	0	1	0	0	1	0	0	1	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

   In our example $f(5) = \{3, 4, 6, 8, 9, \cdots\}$. Following Cantor we construct the subset $\overline{D} = \{n \in \mathbb{N} : \ n \notin f(n)\}$. So $\overline{D} = \{1, 2, 4, 5, 8, \cdots\}$.

   The subset $\overline{D}$ cannot be same as any one of the subsets present in the table. So it has no preimage in $\mathbb{N}$ - a contradiction, as we started with a bijection and the table should exhaust all the subsets of $\mathbb{N}$.

2. Use *diagonalization* to prove that there is no bijection from $\mathbb{N}$ to $[0, 1)$.

   Ans. Any number $x \in [0, 1)$ can be represented as $0.d_1 d_2 d_3 \cdots d_i \cdots$, where $d_i \in \{0, 1, \cdots, 9\}$. The bijection may be represented by the infinite size table as

n	$f(n)$									
	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	$\cdots$
1	0	1	2	3	4	5	6	7	8	$\cdots$
2	1	2	3	4	5	6	7	8	9	$\cdots$
3	2	3	5	5	6	7	8	9	0	$\cdots$
4	3	4	5	6	7	8	9	0	0	$\cdots$
5	**4**	**5**	**6**	**7**	**8**	**9**	**0**	**1**	**2**	$\cdots$
6	0	9	8	7	6	5	4	3	2	$\cdots$
7	3	4	1	5	9	2	6	5	0	$\cdots$
8	2	7	1	8	2	8	1	5	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

So $f(5) = 0.456789012\cdots$.

We construct an $y = 0.D_1D_2D_3\cdots D_i\cdots$, where

$$D_i = \begin{cases} 5 & \text{if } f(i)_i \text{, the } i^{th} \text{ digit of } f(i) \text{, is not 5} \\ 6 & \text{if } f(i)_i \text{, the } i^{th} \text{ digit of } f(i) \text{, is 5} \end{cases}$$

So $y = 0.55655655\cdots$ cannot be same as any value of $f(n)$ present in the table and it does not have any preimage - a contradiction.

3. Use *diagonalization* to prove that the language

$$H = \{< M, x >: \text{ the DTM } M \text{ halts on input } x\}.$$

is not *decidable*(*recursive*).

Ans. Suppose H is decided by the DTM M_H i.e.
M_H:
Input: $< M, x >$
  if M halts on x, then *accept*
  else *reject*.

We define another DTM D as follows:
D:
Input: $< M >$
  if M_H rejects the input $< M, M >$, then *accept*
  f M_H *accepts* $< M, M >$, then *reject*.

We apply D on its own description, $< D >$. The outcome is
$D(< D >)$ *accepts* if and only if $M(< D, D >) \equiv D(< D >)$ *rejects* - a contradiction.

Following table shows the the behavior of M_H on different pairs of $< M_i, M_j >$.

M_i	$< M_j >$									
	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	$\cdots$	D
M_1	R	A	A	R	A	R	R	A	A	$\cdots$	
M_2	R	R	A	R	A	A	R	R	A	$\cdots$	
M_3	R	R	A	R	A	A	R	R	A	$\cdots$	
M_4	A	A	A	R	A	R	R	R	R	$\cdots$	
M_5	R	R	A	A	R	A	R	A	A	$\cdots$	
M_6	R	A	R	R	A	A	A	R	A	$\cdots$	
M_7	A	A	A	R	R	R	A	R	R	$\cdots$	
M_8	A	R	R	A	A	A	A	R	R	$\cdots$	
M_9	R	A	R	R	A	R	R	A	A	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
D	**A**	**A**	**R**	**A**	**A**	**R**	**R**	**A**	**R**	$\cdots$	??