School of Mathematical and Computational Sciences

Indian Association for the Cultivation of Science

Master's/Integrated Master's-PhD Program/ Integrated Bachelor's-Master's Program/PhD Course

Theory of Computation II: COM 5108

Tutorial II (17 August 2023)

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1. What does the TM $M = (\{s, q_0, q_1\}, \{0, 1, \triangleright, \sqcup\}, \delta, s)$ compute on input $\triangleright x$, where $x \in \{0,1\}^+$?

$p \in Q$	$\sigma \in \Sigma$	$\delta(p, \sigma) = (q, \gamma, D)$
s	$\triangleright$	$(s, \triangleright, \rightarrow)$
s	0	$(s, 0, \rightarrow)$
s	1	$(s, 1, \rightarrow)$
s	$\sqcup$	$(q_0, \sqcup, \leftarrow)$
q_0	0	$(q_0, 0, \leftarrow)$
q_0	1	$(q_1, 1, \leftarrow)$
q_0	$\triangleright$	$(h, \triangleright, \rightarrow)$
q_1	0	$(q_1, 1, \leftarrow)$
q_1	1	$(q_1, 0, \leftarrow)$
q_1	$\triangleright$	$(h, \triangleright, \rightarrow)$

Ans. Computes the 2's complement of the given data.

2. (a) Design a single tape Turing machine that computes a function $f : \{0,1\}^* \rightarrow \{0,1\}^*$ i.e. it takes an input $x \in \{0,1\}^*$ and produces an output $f(x) = y \in \{0,1\}^*$ such that each '0' and '1' of x will be replaced by '00' and '11' respectively. As examples, $f(\varepsilon) = \varepsilon$, $f(0) = 00$, $f(1) = 11$, $f(101) = 110011$ etc.

Ans. The TM $M = (Q, \Sigma, \delta, s)$, where

$$Q = \{s, s_0, s_1, q_0, q_1, q_2, q_3\}$$
$$\Sigma = \{0, 1, a, b, \triangleright, \sqcup\}$$

The transition function is as follows:

$p \in Q$	$\sigma \in \Sigma$	$\delta(p,\sigma) = (q,\gamma,D)$
s	$\rhd$	$(s, \rhd, \rightarrow)$
s	$\sqcup$	$(h, \sqcup, \leftarrow)$
s	0	$(s_0, a, \rightarrow)$
s	1	$(s_1, b, \rightarrow)$
s_0	0	$(q_0, a, \rightarrow)$
s_0	1	$(q_1, a, \rightarrow)$
s_0	$\sqcup$	$(q_3, a, -)$
s_1	0	$(q_0, b, \rightarrow)$
s_1	1	$(q_1, b, \rightarrow)$
s_1	$\sqcup$	$(q_3, b, -)$
q_0	0	$(q_0, 0, \rightarrow)$
q_0	1	$(q_1, 0, \rightarrow)$
q_0	$\sqcup$	$(q_2, 0, \leftarrow)$
q_1	0	$(q_0, 1, \rightarrow)$
q_1	1	$(q_1, 1, \rightarrow)$
q_1	$\sqcup$	$(q_2, 1, \leftarrow)$
q_2	0	$(q_2, 0, \leftarrow)$
q_2	1	$(q_2, 1, \leftarrow)$
q_2	a	$(s, a, \rightarrow)$
q_2	b	$(s, b, \rightarrow)$
q_3	a	$(q_3, 0, \leftarrow)$
q_3	b	$(q_3, 1, \leftarrow)$
q_3	$\rhd$	$(h, \rhd, -)$

(b) Show every step of computation (sequence of configurations) on input $\varepsilon, 0, 10, 110$

Ans.

- $(s, \rhd, \varepsilon) \rightarrow (s, \rhd\sqcup, \varepsilon) \rightarrow (h, \rhd, \varepsilon)$.
- $(s, \rhd, 0) \rightarrow (s, \rhd 0, \varepsilon) \rightarrow (s_0, \rhd a\sqcup, \varepsilon) \rightarrow (q_3, \rhd aa, \varepsilon) \rightarrow (q_3, \rhd a, 0) \rightarrow (q_3, \rhd, 00) \rightarrow (h, \rhd, 00)$.
- $(s, \rhd, 10) \rightarrow (s, \rhd 1, 0) \rightarrow (s_1, \rhd b0, \varepsilon) \rightarrow (q_0, \rhd bb\sqcup, \varepsilon) \rightarrow (q_2, \rhd bb0, \varepsilon) \rightarrow (q_2, \rhd bb, 0) \rightarrow (s, \rhd bb0, \varepsilon) \rightarrow (s_0, \rhd bba\sqcup, \varepsilon) \rightarrow (q_3, \rhd bbaa, \varepsilon) \rightarrow (q_3, \rhd bba, 0) \rightarrow (q_3, \rhd bb, 00) \rightarrow (q_3, \rhd b, 100) \rightarrow (q_3, \rhd, 1100) \rightarrow (h, \rhd, 1100)$.
- $(s, \rhd, 110) \rightarrow (s, \rhd 1, 10) \rightarrow (s_1, \rhd b1, 0) \rightarrow (q_1, \rhd bb0, \varepsilon) \rightarrow (q_0, \rhd bb1\sqcup, \varepsilon) \rightarrow (q_2, \rhd bb1, 0) \rightarrow (q_2, \rhd bb, 10) \rightarrow (s, \rhd bb1, 0) \rightarrow (s_1, \rhd bbb0, \varepsilon) \rightarrow (q_0, \rhd bbbb\sqcup, \varepsilon) \rightarrow (q_2, \rhd bbbb, 0) \rightarrow (s, \rhd bbbb0, \varepsilon) \rightarrow (s_0, \rhd bbbba\sqcup, \varepsilon) \rightarrow (q_3, \rhd bbbbaa, \varepsilon) \rightarrow^* (q_3, \rhd, 1100) \rightarrow (h, \rhd, 1100)$.

(c) Compute the number of steps in terms $|x| = n$ as accurately as you can.

Ans. We take a longer string and process the leftmost bit.

$$
\begin{aligned}
& (s, \rhd, 1010101010) \\
\to\ & (s, \rhd 1, 010101010) \\
\to\ & (s_1, \rhd b0, 10101010) \\
\to\ & (q_0, \rhd bb1, 0101010) \\
\to\ & (q_1, \rhd bb00, 101010) \\
\to\ & (q_0, \rhd bb011, 01010) \\
\to\ & (q_1, \rhd bb0100, 1010) \\
\to\ & (q_0, \rhd bb01011, 010) \\
\to\ & (q_1, \rhd bb010100, 10) \\
\to\ & (q_0, \rhd bb0101011, 0) \\
\to\ & (q_1, \rhd bb01010100, \varepsilon) \\
\to\ & (q_0, \rhd bb01010101\sqcup, \varepsilon) \\
\to\ & (q_2, \rhd bb01010101, 0) \\
\to\ & (q_2, \rhd bb0101010, 10) \\
\to\ & (q_2, \rhd bb010101, 010) \\
\to\ & (q_2, \rhd bb01010, 1010) \\
\to\ & (q_2, \rhd bb0101, 01010) \\
\to\ & (q_2, \rhd bb010, 101010) \\
\to\ & (q_2, \rhd bb01, 0101010) \\
\to\ & (q_2, \rhd bb0, 10101010) \\
\to\ & (q_2, \rhd bb, 010101010) \\
\to\ & (s, \rhd bb0, 10101010)
\end{aligned}
$$

So the number of steps are $2(n+\overline{n-1}+\cdots+2+1)+2n+C = O(n^2)$.

(d) What is the time complexity?

3. (a) Design a 2-tape Turing machine for the language of **(??)**. The output will be on the second tape. Clearly specify the start and end configurations.

Ans. The 2-tape TM is $M = (\{s, t_0, t_1\}, \{0, 1, \rhd, \sqcup\}, \delta, s)$. The state transition function δ is as follows.

$p \in Q$	$\sigma_1 \in \Sigma$	$\sigma_2 \in \Sigma$	$\delta(p, \sigma_1, \sigma_2) = (q, \gamma_1, D_1, \gamma_2, D_2)$
s	$\rhd$	$\rhd$	$(s, \rhd, \rightarrow, \rhd, \rightarrow)$
s	$\sqcup$	$\sqcup$	$(h, \sqcup, -, \sqcup, -)$
s	0	$\sqcup$	$(t_0, 0, -, 0, \rightarrow)$
t_0	0	$\sqcup$	$(s, 0, \rightarrow, 0, \rightarrow)$
s	1	$\sqcup$	$(t_1, 1, -, 1, \rightarrow)$
t_1	1	$\sqcup$	$(s, 1, \rightarrow, 1, \rightarrow)$

The input configuration is $(s, \rhd, x, \rhd, \varepsilon)$.

(b) Show every step of computation on input $\varepsilon, 0, 10, 110$

Ans.

- $(s, \triangleright, \varepsilon, \triangleright, \varepsilon) \to (s, \sqcup, \varepsilon, \sqcup, \varepsilon) \to (h, \sqcup, \varepsilon, \sqcup, \varepsilon)$
- $(s, \triangleright, 0, \triangleright, \varepsilon) \to (s, 0, \varepsilon, \sqcup, \varepsilon) \to (t_0, 0, \varepsilon, 0\sqcup, \varepsilon) \to (s, 0\sqcup, \varepsilon, 00\sqcup, \varepsilon) \to (h, 0\sqcup, \varepsilon, 00\sqcup, \varepsilon)$
- $(s, \triangleright, 10, \triangleright, \varepsilon) \to (s, 1, 0, \sqcup, \varepsilon) \to (t_1, 1, 0, 1\sqcup, \varepsilon) \to (s, 10, \varepsilon, 11\sqcup, \varepsilon) \to (t_0, 10, \varepsilon, 110\sqcup, \varepsilon) \to (s, 10\sqcup, \varepsilon, 1100\sqcup, \varepsilon) \to (h, 10\sqcup, \varepsilon, 1100\sqcup, \varepsilon)$

(c) Compute the number of steps in terms $|x| = n$ as accurately as you can.

Ans. The number of steps are $2(n+1)$.

(d) What is the time complexity?

Ans. $O(n)$.