School of Mathematical and Computational Sciences

# Indian Association for the Cultivation of Science

*Master's/Integrated Master's-PhD Program/ Integrated Bachelor's-Master's Program/PhD Course*

## Theory of Computation II: COM 5108

*Quiz II (12 October 2023)*

## Answer All Questions

**Marks: (1+2) + (1+1) + 4 + (4 + 2) = 15**

1. A language $L$ is **PSPACE**-complete.

   (i) Is $L$ **NP**-hard?

   (ii) Is $L$ **NP**-complete?

   Justify your answers and assumption.

   **Ans.**

   (i) It is known that $\mathbf{NP} \subseteq \mathbf{PSPACE}$. So all $L' \in \mathbf{NP}$ is also in **PSPACE** and $L' \leq_p L$. So $L$ is **NP**-hard.

   (ii) The answer to this part is unknown as it is unknown whether **NP** and **PSPACE** are the same. If they are equal, people believe it to be unlikely, then the statement is true, otherwise $L$ cannot belong to **NP**, and the statement is false. If it is known that $\mathbf{P} \neq \mathbf{PSPACE}$ and a **PSPACE**-complete problem $L$ is in **NP**, then every problem of **PSPACE** will also be in **NP** - a contradiction.

2. Does $\phi$ and/or $\psi$ belong to TQBF?

   (i) $\psi = \exists b \forall a \exists c (a \vee b) \wedge (\overline{a} \vee c) \wedge (\overline{b} \vee \overline{c})$.

   (ii) $\phi = \forall a \exists b \exists c (a \vee b) \wedge (\overline{a} \vee c) \wedge (\overline{b} \vee \overline{c})$.

   **Ans.**

   (i) $\psi \notin TQBF$ as $b = 0, a = 0$ makes it *false* (0). And also $b = 1, a = 1$ makes it *false* (0).

   (ii) $\phi \in TQBF$ as both $a = 0, b = 1, c = 0$ and $a = 1, b = 0, c = 1$ make it *true* (1).

3. Consider the CFG $G = (\{0,1\}, \{S\}, R, S)$, where the production rules are $S \rightarrow \varepsilon \mid 0\,S\,1 \mid SS$.
Is $L(G)$, the language of $G$, in **L**?
Clearly justify your answer.

   **Ans.** $L(G) \in \mathbf{L}$. The following logspace bounded TM decides it

   $M$ : Input $< G, s, d >$
   1 $c = 0$
   2 do 3 to 5 until the end of the input
   3 if the input is 0 $c \leftarrow c + 1$
   4 if the input is 1 $c \leftarrow c - 1$
   5 if $c < 0$, halt with '**No**'
   6 if $c = 0$, halt with '**Yes**'
   7 otherwise halt with '**No**'

   The length of the counter is $\lceil \log_2 |x| \rceil$. So the answer.

4. A GG game is a 2-player ($\{I, II\}$) game played as follows. $G$ is a directed graph with a designeted start node $s$.

   (i) The player '$I$' starts the game from the start node $s$. Each player gives alternate moves.

   (ii) A move by a player is to pick a new node in the graph on a *simple directed path* from the current node. A *simple path* is one where no node has already been visited.

   (iii) A player loses if she fails to make a move.

   $GG = \{< G, s >$: player '$I$' has a winning strategy on the directed graph $G$, starting from $s\}$.

   (a) Following algorithm (incomplete) decides GG. **Fill-in the blanks** to complete the algorithm. Give proper justifications.

   $M$ : Input $< G, s >$
   If *out-degree* of $s$ is zero, then $\cdots (i) \cdots$-halt
   Remove $s$ and all edges in and out of it. The new graph is $G'$ where $s_1, s_2, \cdots, s_k$ are nodes pointed by $s$ in $G$.
   Give recursive calls to $M$ with parameters $\cdots (ii) \cdots$.
   If all these calls reach '**Yes**'-*halt*, then $\cdots (iii) \cdots$ - halt.
   Otherwise, $\cdots (iv) \cdots$ - halt.

   (b) Justify that $GG \in$ **PSPACE**.

   **Ans.**

   (a) (i) '**No**'-halt.
   (ii) $< G', s_1 >, < G', s_2 >, \cdots, < G', s_k >$.
   (iii) '**No**'-halt.
   (iv) '**Yes**-halt.

   (b) Each call passes the modified graph and the node. The depth of the call is at most the number of nodes. So the space usage is quadratic. The algorithm is in **PSPACE**.