# Indian Association for the Cultivation of Science

## (Deemed to be University under *de novo* Category)

*Master's/Integrated Master's-PhD Program/ Integrated Bachelor's-Master's Program/PhD Course*

*Mid-Semester (Sem-III) Examination-Autumn 2023*

*Subject:* **Theory of Computation II** *Subject Code:* **COM 5108‘**

*Full Marks:* **25** *Time Allotted:* **2 hours**

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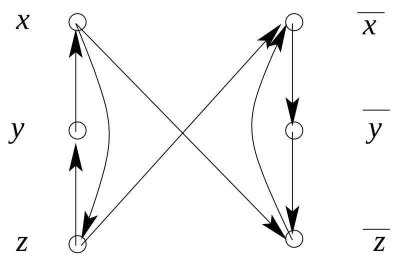
**Q 1. Answer five (5) questions with brief justifications. [5 × 2 = 10]**

(a) Is $2^n = o(3^n)$ (small 'o')?

**Ans.** Yes. $\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$.

(b) Draw the implication graph (directed) corresponding to the 2SAT formula $\phi = (x \vee \overline{y}) \wedge (y \vee \overline{z}) \wedge (z \vee \overline{x}) \wedge (\overline{z} \vee \overline{x})$. Justify from the graph why all variables get the truth value *false* for a *satisfying* assignment.

**Ans.** The implication graph is as follows.




No variable can get the truth value *true* as for each variable there is a path from the variable to its negation.

(c) An NTM has a 3-way nondeterminism at the given state-input pair: $\delta(q_0, 0) = \{(q_1, 0, \leftarrow), (q_2, 0, -), (q_3, 1, \rightarrow)\}$. Modify the transition function so that there will be 2-way nondeterminism.

**Ans.** Let the modified transition function be $\delta'$. It is same as $\delta$ for all state-input pairs other than $(q_0, 0)$. Let $p$ be a new state, the transitions are $\delta'(q_0, 0) = \{(q_1, 0, \leftarrow), (p, 0, -)\}$ and $\delta'(p, 0) = \{(q_2, 0, -), (q_3, 1, \rightarrow)\}$.

(d) The language $coFin = \{L \subseteq \Sigma^* : \Sigma^* \setminus L \text{ is finite}\}$. Is $coFin \in \mathbf{P}$?

**Error.** it is not '$\in$' but '$\subseteq$'.

**Ans.** Every finite language is *regular*. So every *coFin* language is also regular. And a *regular* language is also in **P**.

(e) Is the following language in **P** or **NP**-hard?

$$SAT_e = \{\phi\, 0\, 1^{2^n} : \phi \in SAT \text{ and } |\phi| = n\}.$$

**Ans.** Identification of the formula $\phi$ can be done in linear time. The membership of $\phi$ in SAT can be tested using a truth table method. The size of the truth table is $O(2^n)$. $2^n$ can also be computed in time $O(2^n)$ and the number of 1's at the end of the input can be checked. So time taken is linear in the length of the input and $SAT_e \in \mathbf{P}$.

(f) What is the total number of variable instances used in this formula where $C = Q \cup \Gamma$ and $|C| = l$.

$$\phi_{cell} = \bigwedge_{1\leq i,j\leq n^k} \left( \left( \bigvee_{p\in C} x_{i,j,p} \right) \wedge \left( \bigwedge_{\substack{p,q\in C,\\ p\neq q}} (\overline{x_{i,j,p}} \vee \overline{x_{i,j,q}}) \right) \right).$$

**Ans.** $\bigvee_{p\in C} v_{i,j,p}$ uses $l$ variables. $\bigwedge_{\substack{p,q\in C,\\ p\neq q}} (\overline{v_{i,j,p}} \vee \overline{v_{i,j,q}})$ uses $2\times\binom{l}{2} = l(l-1)$ variables. So the total number of variables used are $(n^k)^2 \times l^2$.

(g) A language $L \in \mathbf{NP}$ is defined as follows.

$\forall x \in \Sigma^*$, $x \in L$ if and only if $\exists w \in \Sigma^{p(n)}$, there is a $q(n)$ time bounded DTM $V$ that accepts $< x, w >$,

where $n = |x|$, $p(n)$, $q(n)$ are polynomials.
Define a language $L' \in \mathbf{coNP}$ in a similar way.

**Ans.** Language $L' \in \mathbf{coNP}$ is defined as follows.

$\forall x \in \Sigma^*$, $x \in L'$ if and only if $\forall w \in \Sigma^{p(n)}$, there is a $q(n)$ time bounded DTM $\neg V$ that accepts $< x, w >$,

where $n = |x|$, $p(n)$, $q(n)$ are polynomials, and $\neg V$ is same as $V$ with *accept* and *reject* states reversed.

## Answer any three (3) of the following questions.

**[3 × 5 = 15]**

**Q 2.** Give an $O(n \log n)$ step bounded single-tape DTM algorithm to recognize $L = \{x \in \{0,1\}^* : x$ has equal number of 0's and 1's$\}$, where $n$ is the length of the input. Detail state transitions are not required, but explain its operation and the time complexity.

**Ans.** $M =$ "On input $x$

1. Scan the string and put a *sentinel* at the end of the input - $O(n)$ steps.
2. Repeat the following steps as long as there are some 0's and 1's.
   (i) Check the parity of the remaining 0's and 1's. If it is *odd* **reject** - $O(n)$ steps.
   (ii) Replace alternate 0's by '$\sqcap$' starting with the first one. Do the same thing for 1's - $O(n)$ steps.
3. **Accept**.

Step (1) takes $O(n)$ steps. The loop runs for $\log n$ times. Each of $(i)$ and $(ii)$ take $O(n)$ steps. The total number of steps are $O(n) + O(\log n)(O(n) + O(n)) = O(n \log n)$.

**Q 3.** Give the detail design of a DTM $M = (Q, \Sigma, \delta, s)$, that takes the input $\rhd x$, where $x \in \{0,1\}^+$ and computes the 2's complement of $x$ (ignore overflow). Examples of computation are: $\rhd 0 \to_M^* \rhd 0$, $\rhd 1 \to_M^* \rhd 0$, $\rhd 101100 \to_M^* \rhd 010100$.

**Ans.** The TM $M = (\{s, q_0, q_1\}, \{0, 1, \rhd, \sqcup\}, \delta, s)$ where the state transition is given by the following table.

| $p \in Q$ | $\sigma \in \Sigma$ | $\delta(p, \sigma) = (q, \gamma, D)$ |
|---|---|---|
| $s$ | $\rhd$ | $(s, \rhd, \rightarrow)$ |
| $s$ | 0 | $(s, 0, \rightarrow)$ |
| $s$ | 1 | $(s, 1, \rightarrow)$ |
| $s$ | $\sqcup$ | $(q_0, \sqcup, \leftarrow)$ |
| $q_0$ | 0 | $(q_0, 0, \leftarrow)$ |
| $q_0$ | 1 | $(q_1, 1, \leftarrow)$ |
| $q_0$ | $\rhd$ | $(h, \rhd, \rightarrow)$ |
| $q_1$ | 0 | $(q_1, 1, \leftarrow)$ |
| $q_1$ | 1 | $(q_1, 0, \leftarrow)$ |
| $q_1$ | $\rhd$ | $(h, \rhd, \rightarrow)$ |

**Q 4.** $3COLOR = \{< G >: G$ is an undirected graph whose vertices can be coloured with at most 3 colours $\}$. Give a polynomial time reduction of $3COLOR$ to $SAT$, $G \mapsto \phi$, $G$ is *3-colourable* if and only if $\phi$ is *satisfiable*.

**Ans.** Let the graph $G = (V, E)$ has $n$ vertices $V = \{v_1, \cdots, v_n\}$. We take $3n$ variables, $\{x_{11}, x_{12}, x_{13}, \cdots, x_{n1}, x_{n2}, x_{n3}\}$. The variable $x_{ij},\ 1 \leq i \leq n,\ 1 \leq j \leq 3$, is *true* if the vertex $v_i$ is coloured with the colour $j$. We have the following set of clauses:

(a) Each vertex must have at least one colour:

$$\bigwedge_{i=1}^{n} (x_{i1} \vee x_{i2} \vee x_{i3}).$$

(b) No vertex can have two colours:

$$\bigwedge_{i=1}^{n} \bigwedge_{1 \leq j \neq k \leq 3} (\neg x_{ij} \vee \neg x_{ik})$$

(c) No pair of adjacent vertices can have same colour.

$$\bigwedge_{\{v_a, v_b\} \in E} \bigwedge_{j=1}^{3} (\neg x_{aj} \vee \neg x_{bj}).$$

The length is of $O(n) + O(n) + O(n^2) = O(n^2)$.

**Q 5.** Informally describe how an $f(n)$ time bounded $k$-tape DTM can be simulated on a single-tape DTM. What is the order of *slowdown* of the simulation?