Countable, Denumerable and Finite Sets

Countable Set

A set A is said to be **countable**, if there is an *injection* or *one-to-one* map from A to the set of natural numbers, $\mathbb{N}$ $(\{0, 1, 2, \cdots\})$.

Examples

- The *null set* is countable as the only function is vacuously an injection.
- Any set with finite number of elements (n elements) is countable. The ith element will be mapped to i, $0 \leq i < n$.
- Any subset B of $\mathbb{N}$ is countable. The *inclusion map* $i : B \longrightarrow \mathbb{N}$, $i(k) = k$ is an injection.
- Are the sets of *real numbers* or *complex numbers* countable?

Infinite Set

A set A is said to be an **infinite** set, if there is an *injection* or *one-to-one* map from the set of natural numbers $\mathbb{N}$, to the set A.

Examples

- The set of *natural numbers* is an *infinite set*. The identity map is an *injection*.
- The set of even numbers is an infinite set. We already have an injection $f : \mathbb{N} \longrightarrow \mathbb{E}$, $f(n) = 2n$.
- The collection of all *prime numbers* is an infinite set. It is more tricky to show that there is an *injection*.
- The set of *real numbers*, $\mathbb{R}$ is an infinite set. The *inclusion map* $i : \mathbb{N} \longrightarrow \mathbb{R}$, $i(n) = n$ is an *injection*.
- How do you show that the set of *rational* or *complex numbers* are infinite sets?

Denumerable and Finite Sets

- A set A is said to be *denumerable*, if it is both *countable* and *infinite* i.e. there is an *injection* from A to $\mathbb{N}$ and also there is an *injection* from $\mathbb{N}$ to A.
- A set A is said to be *finite*, if it is not *infinite* i.e. there is no *injection* from $\mathbb{N}$ to A.

Examples : Denumerable Sets

- The set of *natural numbers* is *denumerable*. The identity map is an *injection*.
- The set of even numbers is *denumerable*. We already have an *injection* $f : \mathbb{N} \longrightarrow \mathbb{E}$, $f(n) = 2n$. The inverse of f is an *injection* in the other direction.
- The collection of all *prime numbers* is also *denumerable*.
- Are the set of *rational*, *real* or *complex numbers* *denumerable*?

Observe

- A is *countable* : $A \leq \mathbb{N}$.
- A is *infinite* : $\mathbb{N} \leq A$.
- A is *denumerable* : $A \leq \mathbb{N}$ and $\mathbb{N} \leq A$. [We are not equipped to say that $A \simeq \mathbb{N}$.]
- A is *finite* : $\mathbb{N} \not\leq A$.

Schröder - Bernstein Theorem

For two sets A and B, $A \leq B$ and $B \leq A$ implies $A \simeq B$

- If the *cardinality* of A is $\leq$ the *cardinality* of B and also the *cardinality* of B is $\leq$ the *cardinality* of A, then A and B are *equinumerous* (of same *cardinality*).
- In other words, if there are *injections* $f : A \longrightarrow B$ and $g : B \longrightarrow A$, there is a *bijection* from A to B.

To show that two sets are of same *cardinality*, it is often easy to construct a pair of *injections* than to construct a *bijection*.

A Beautiful Proof (*Dedekind*?)

A notation: Let $h : B \longrightarrow A$ be a map and $C \subseteq A$. We define

$$h^{-1}(C) = \{b \in B : h(b) \in C\}$$

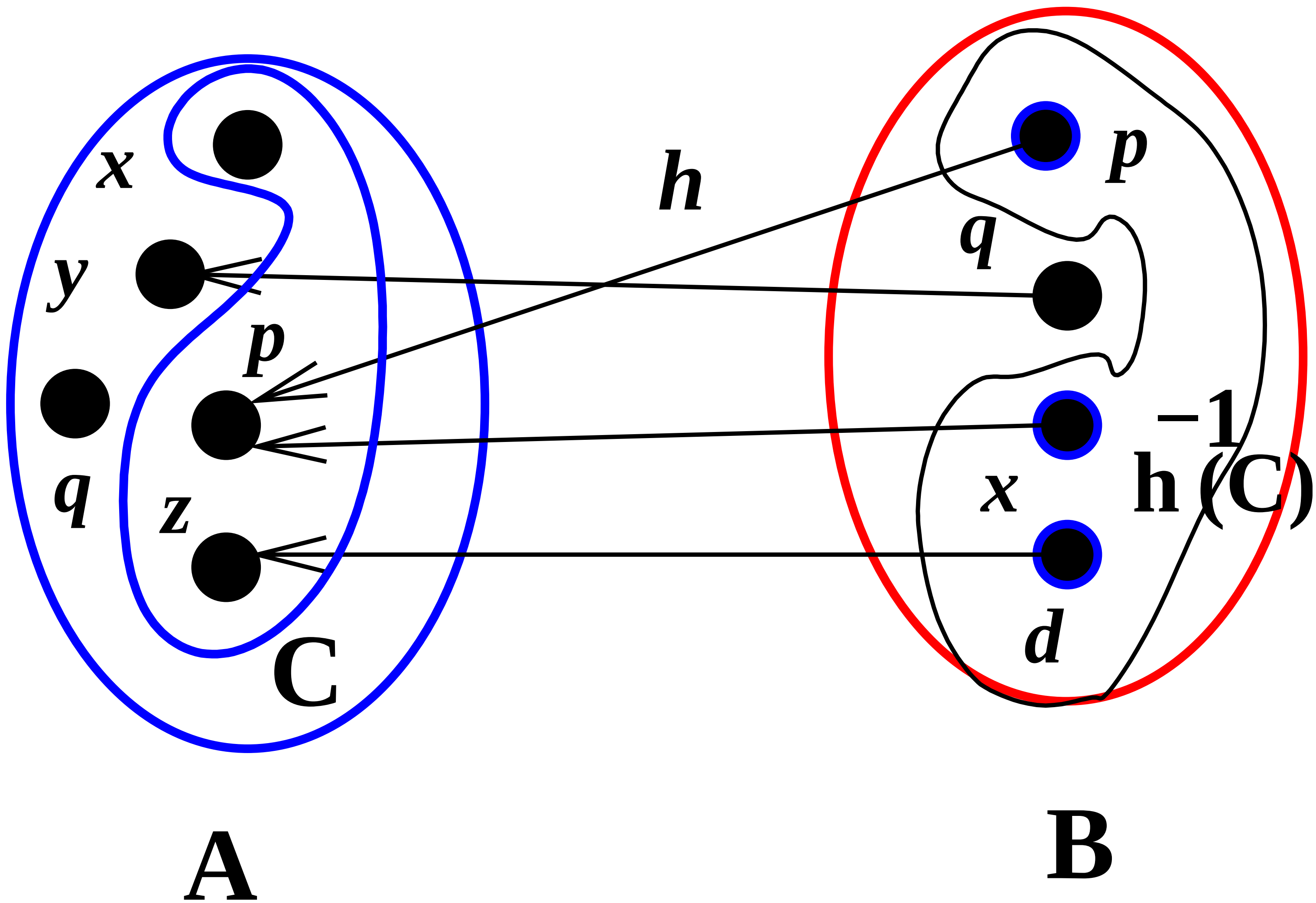

Figure 1: $h^{-1}(C)$

Proof (*cont.*)

- Given two *one-to-one* maps $f : A \longrightarrow B$ and $g : B \longrightarrow A$, there exists $C \subseteq A$ such that $g^{-1}(A \setminus C) = B \setminus f(C)$.

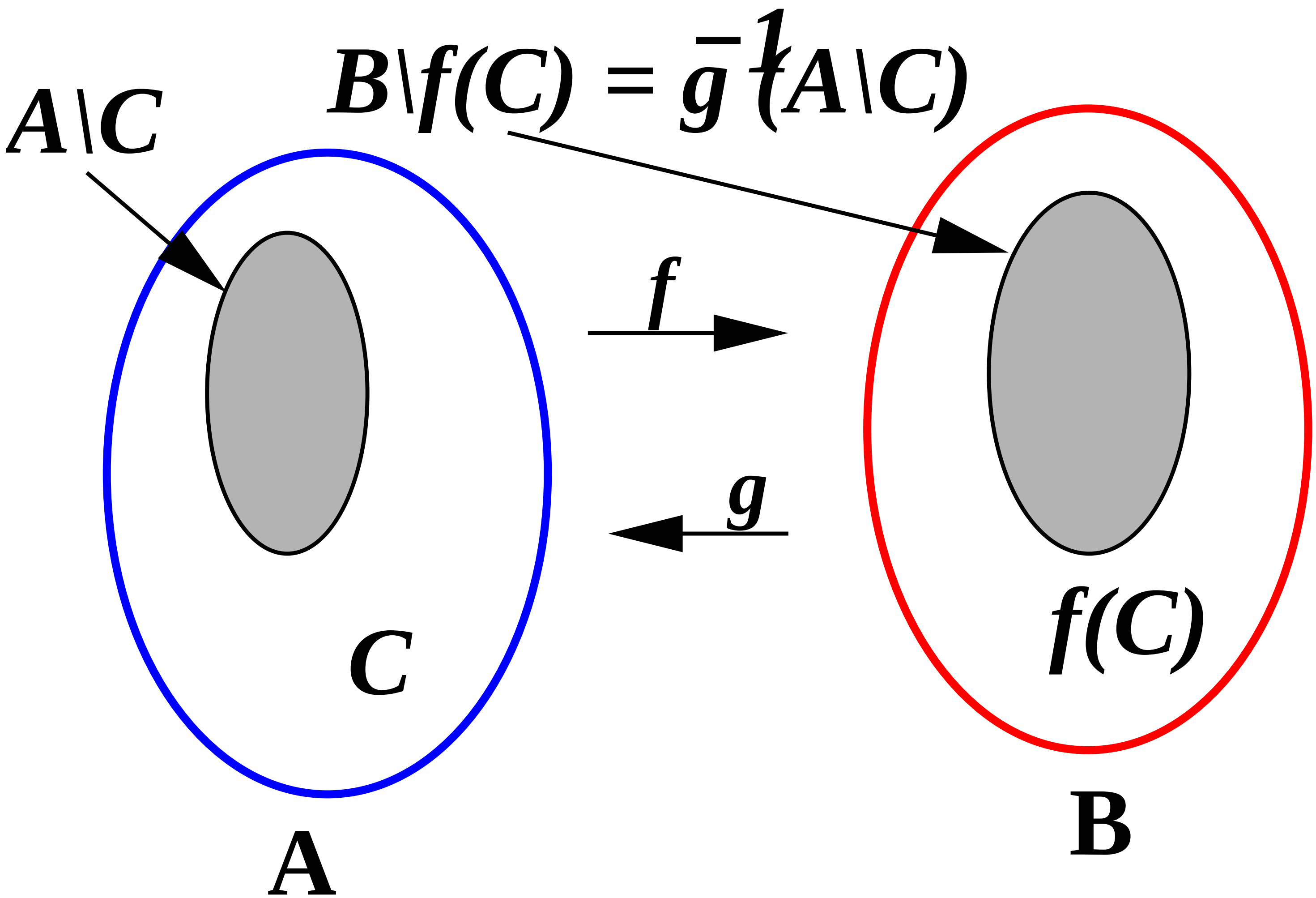

Figure 2: $g^{-1}(A \setminus C) = B \setminus f(C)$

Proof (*cont.*)

- We define the function $F : \mathcal{P}A \longrightarrow \mathcal{P}A$ on the *power set* of A using f and g.

$$F(C) = A \setminus g(B \setminus f(C)), \text{ for all } C \in \mathcal{P}A.$$

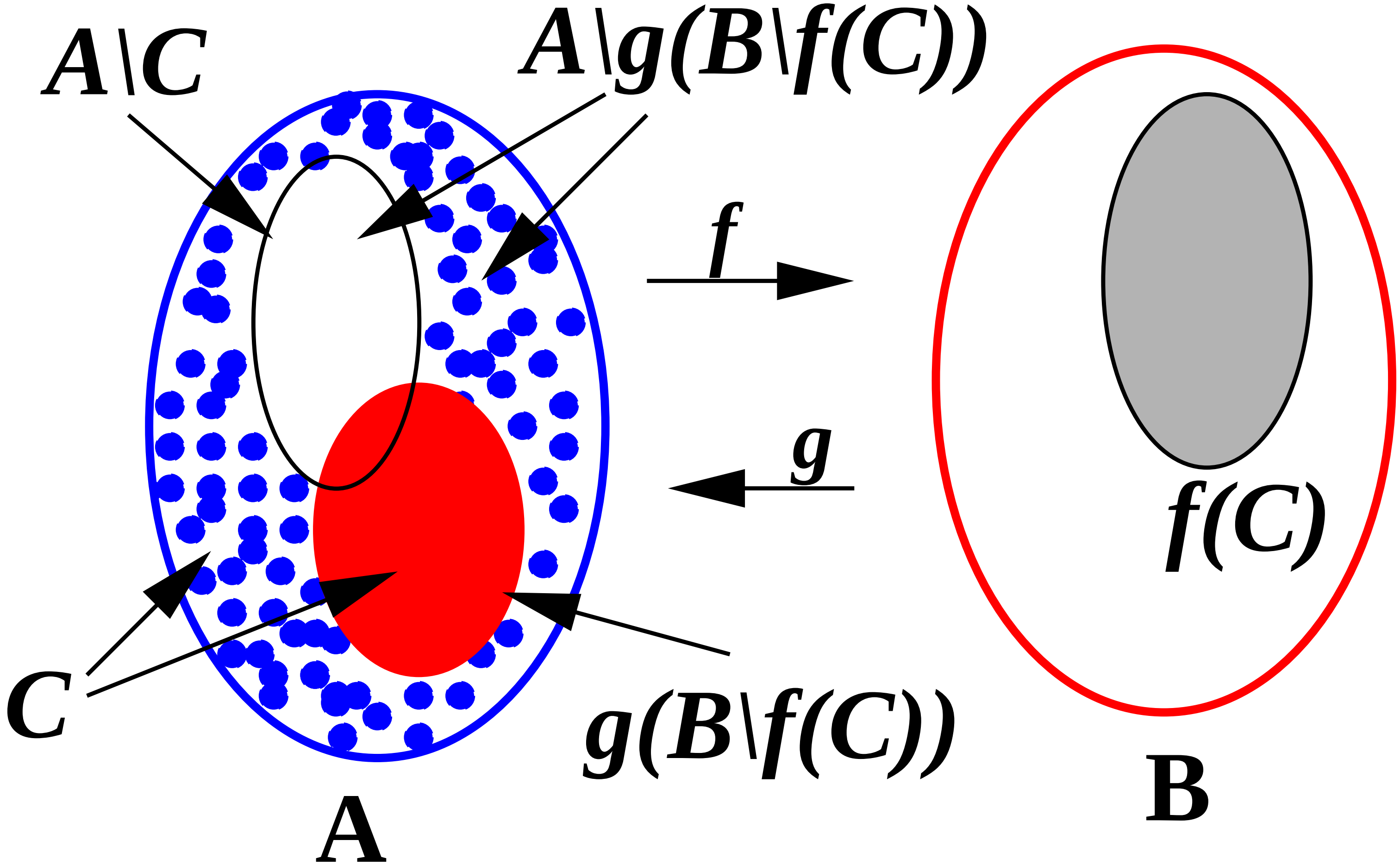

Figure 3: $F(C) = A \setminus g(B \setminus f(C))$

Proof (*cont.*)

- It can be proved that F is *monoton* i.e. $X \subseteq Y \subseteq A$ imples that $F(X) \subseteq F(Y)$, and
- It can also be proved that F is *continuous* i.e. if $\{A_i\}_{i \in I}$ be a collection of subsets the of A, then

$$F(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} F(A_i).$$

These two conditions and the structure of $\mathcal{P}A$ will guarantee the existence of a *fixed point* of F.

Fixed Point of a Function

A *fixed point* of a function $f : A \longrightarrow A$ is an element $a \in A$, such that $f(a) = a$.

- $s : \mathbb{N} \longrightarrow \mathbb{N}$, so that $s(n) = n + 1$, does not have any fixed point.
- $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, so that $f(k) = 4 - k$, has exactly one *fixed point* at $k = 2$.
- $1_{\mathbb{N}} : \mathbb{N} \longrightarrow \mathbb{N}$, so that $1_{\mathbb{N}}(n) = n$, has infinitely many *fixed points*.

Proof (*cont.*)

Consider the collection of subsets of A, $\{A_i\}_{i\in\mathbb{N}}$, defined inductively as follows.

$$A_i = \begin{cases} \emptyset & \text{if } i = 0, \\ F(A_{i-1}) & \text{otherwise.} \end{cases}$$

Let $C = \bigcup_{i\in\mathbb{N}}\{A_i\}$. It is not difficult to prove that $F(C) = C$, a *fixed point* of F and this is the desired C.

$$\begin{aligned} F(C) = A \setminus g(B \setminus f(C)) = C \quad &\Rightarrow \quad A \setminus C = g(B \setminus f(C)) \\ &\Rightarrow \quad g^{-1}(A \setminus C) = B \setminus f(C). \end{aligned}$$

Example

Consider $f : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$, and $g : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$, so that $f(n) = (n, n)$ and $g(n, m) = 2^n 3^m$. We can prove that both f and g are *one-to-one* maps.

We have $F : \mathcal{P}\mathbb{N} \longrightarrow \mathcal{P}\mathbb{N}$, and
$F(C) = \mathbb{N} \setminus g((\mathbb{N} \times \mathbb{N}) \setminus f(\mathbb{N}))$.

- $A_0 = \emptyset$.

Example

$$
\begin{aligned}
A_1 &= \mathbb{N} \setminus g((\mathbb{N} \times \mathbb{N}) \setminus f(\emptyset)) \\
&= \mathbb{N} \setminus g(\mathbb{N} \times \mathbb{N}) \\
&= \mathbb{N} \setminus \left\{ \begin{array}{cccc} 1, & 3, & 9, & \cdots \\ 2, & 6, & 18, & \cdots \\ 4, & 12, & 36, & \cdots \\ & \cdots & & \end{array} \right\} \\
&= \{0, 5, 7, 10, 11, 13, \cdots\}
\end{aligned}
$$

Proof (*cont.*)

Using the $C \subseteq A$ that satisfies $g^{-1}(A \setminus C) = B \setminus f(C)$, we define a function $h : A \longrightarrow B$, such that

$$h(a) = \begin{cases} f(a) & \text{if } a \in C, \\ b & \text{if } a \in A \setminus C \text{ and } g(b) = a. \end{cases}$$

The map h is a *bijection*.

Is h a Bijection?

- h is a *one-to-one* : f is *one-to-one* and it maps C to $f(C)$. Each element of $A \setminus C$ is a *g-image* of some element of $B \setminus f(C)$ because we know

$$C = A \setminus g(B \setminus f(C)) \;\Rightarrow\; A \setminus C = g(B \setminus f(C))$$

Again g is *one-to-one*, hence for each $a \in A \setminus C$ there is exactly one $b \in B \setminus f(C)$.

Is h a Bijection (*cont.*)?

- h is *onto* B : B is divided in two parts. Images of C under f, $f(C)$ and $B \setminus f(C)$ covered by invese images of g.

F is Monoton

$F : \mathcal{P}A \longrightarrow \mathcal{P}A$, $F(X) = A \setminus g(B \setminus f(X))$, $X \subseteq A$. Let $X \subseteq Y \subseteq A$.

$$\begin{aligned} X \subseteq Y &\Rightarrow f(X) \subseteq f(Y), \\ &\Rightarrow B \setminus f(Y) \subseteq B \setminus f(X), \\ &\Rightarrow g(B \setminus f(Y)) \subseteq g(B \setminus f(X)), \\ &\Rightarrow A \setminus g(B \setminus f(X)) \subseteq A \setminus g(B \setminus f(Y)). \end{aligned}$$

F is *monoton*.

F is Continuous

Let $\{A_i\}_{i\in I}$ be a collection of the subsets of A.

$$
\begin{aligned}
F(\bigcup\{A_i\}_{i\in I}) &= A \setminus g(B \setminus f(\bigcup\{A_i\}_{i\in I})) \\
&= A \setminus g(B \setminus \bigcup\{f(A_i)\}_{i\in I}) \\
&= A \setminus g(\bigcap\{B \setminus f(A_i)\}_{i\in I}) \\
&= A \setminus \bigcap\{g(B \setminus f(A_i))\}_{i\in I} \\
&= \bigcup\{A \setminus g(B \setminus f(A_i))\}_{i\in I} \\
&= \bigcup\{F(A_i)\}_{i\in I}
\end{aligned}
$$

Equinumerous Sets

- Two finite sets with equal number of elements are *equinumerous*. If there are n elements, $n!$ *bijections* are possible.
- The set of *even* ($\mathbb{E}$) and *odd* ($\mathbb{O}$) non-negative integers are *equinumerous* to the set of natural numbers ($\mathbb{N}$) and therefore to themselves. Following functions are *bijections*.

$$f : \mathbb{E} \longrightarrow \mathbb{N},\ f(2k) = k;\ g : \mathbb{O} \longrightarrow \mathbb{N},\ g(2k+1) = k;$$

$$h : \mathbb{E} \longrightarrow \mathbb{O},\ h(2k) = 2k + 1.$$

Equinumerous Sets (*cont.*)

- The set of integers ($\mathbb{Z}$) is *equinumerous* to the set of natural numbers.

$$f_1 : \mathbb{N} \longrightarrow \mathbb{Z},\ f_1(n) = \begin{cases} n/2 & \text{if } n = 2k, \\ -(n+1)/2 & \text{otherwise.} \end{cases}$$

Equinumerous Sets (*cont.*)

- The set $\mathbb{N} \times \mathbb{N}$ is *equinumerous* to N!

$$\{0, 1, 2, 3, \cdots\} \simeq \left\{ \begin{array}{lll} (0,0), & (0,1), & (0,2), (0,3), \cdots \\ (1,0), & (1,1), & (1,2), (1,3), \cdots \\ (2,0), & (2,1), & (2,2), (2,3), \cdots \\ \quad\vdots & & \end{array} \right\}$$

$f_2 : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$, $f_2(m, n) = (2m + 1)2^n - 1$ is a *bijection*.

Product Map

Two maps $f : A \longrightarrow B$ and $g : C \longrightarrow D$ can induce a map, we call it $f \times g$ (it is a notation), from $A \times C$ to $B \times D$ in the following way.

$$f \times g : A \times C \longrightarrow B \times D, \; f \times g(a, c) = (f(a), g(c)) \in B \times D.$$

Equinumerous Sets (*cont.*)

- The set $\mathbb{Z} \times \mathbb{Z}$ is *equinumerous* to N.
  - $f_1 : \mathbb{N} \longrightarrow \mathbb{Z}$ is a *bijection*.
  - So $f_1^{-1} : \mathbb{Z} \longrightarrow \mathbb{N}$ is also a bijection.
  - We can be prove that $f_1^{-1} \times f_1^{-1} : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{N} \times \mathbb{N}$ is also a *bijection*.
  - $f_2 \circ ({}_1^{-1} \times f_1^{-1}) : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{N}$ is also a *bijection*.

  Let us call $f_2 \circ ({}_1^{-1} \times f_1^{-1})$ as f_3.

Equinumerous Sets (*cont.*)

- The set of *rational numbers* ($\mathbb{Q}$) is *equinumerous* to $\mathbb{N}$.
- The *inclusion map* $i_{\mathbb{Q}} : \mathbb{Q} \longrightarrow \mathbb{Z} \times \mathbb{Z}$, so that $i_{\mathbb{Q}}(\frac{p}{q}) = (p, q)$ is an *injection*.
- $f_3 \circ i_{\mathbb{Q}} : \mathbb{Q} \longrightarrow \mathbb{N}$ is an *injection*.
- $i_{\mathbb{N}} : \mathbb{N} \longrightarrow \mathbb{Q}$, so that $i_{\mathbb{N}}(n) = \begin{cases} 0 & \text{if } n = 0, \\ \frac{n}{1} & \text{otherwise,} \end{cases}$ is also an *injection*.
- By *Schörder-Bernstein* theorem there is a *bijection* from $\mathbb{Q}$ to $\mathbb{N}$.

Equinumerous Sets (*cont.*)

- The set $\overbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}^{k}$ is *equinumerous* to $\mathbb{N}$.
- The *inclusion map* $i_{\mathbb{N}} : \mathbb{N} \longrightarrow \overbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}^{k}$, so that $i_{\mathbb{N}}(n) = (n, \overbrace{0, \cdots, 0}^{k-1})$ is an *injection*.
- The map $f_4 : \overbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}^{k} \longrightarrow \mathbb{N}$, so that $f_4(n_0, \cdots, n_{k-1}) = 2^{n_0} \times 3^{n_1} \times \cdots \times p_{k-1}^{n_{k-1}}$, where '2' is the 0*th*, '3' is the 1*st* and p_{k-1} is the $(k-1)$*th* prime number, is also an *injection*.
- So there is a *bijection* by *Schörder-Bernstein* theorem.

Equinumerous Sets (*cont.*)

- The collection of all finite subsets of natural numbers, $\mathcal{P}_{fin}\mathbb{N}$ is *equinumerous* to $\mathbb{N}$.
- $f_5 : \mathbb{N} \longrightarrow \mathcal{P}_{fin}\mathbb{N}$, so that $f_5(n) = \{n\}$, is an *one-to-one* map.
- $f_6 : \mathcal{P}_{fin}\mathbb{N} \longrightarrow \mathbb{N}$, so that $f_6(\{a_0, a_1, \cdots, a_n\}) = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_n}$, is also an *injection*.
- So there is a *bijection* by *Schörder-Bernstein* theorem.

Alphabet

An *alphabet* is a finite or finitely generated collection of *atomic* symbols.

- The alphabet of decimal (*Arabic-Indian*) numeral : $\{0, 1, \cdots, 9\}$.
- The alphabet for elementary arithmetic : $\{0, 1, \cdots, 9, +, -, \times, /,), (\}$.
- The English language the *alphabet* consists of $a, b, \cdots, z, A, B, \cdots, Z, 0, 1, \cdots, 9$, differnt punctuation marks including 'blank space' and few other symbols.
- Alphabet for C Programming language is almost similar to the English alphabet.

Strings over an Alphabet

The total collection of finite length words using symbols of an *alphabet* (over an *alphabet*) Σ is denoted by Σ^*. This collection includs a special string called the *null string* (ε).

- $\Sigma = \{a, b\}$:
  $\Sigma^* = \{\varepsilon, a, b, aa, abba, bb, aaa, aab, aba, abb, \cdots\}$.

Language over an Alphabet

A *language* L over an *alphabet* Σ is a subset of Σ^*.

- $\Sigma = \{0, 1\}$: Unsigned-binary representation of even numbers without leading zeros is a the language

$$\{0, 10, 100, 110, 1000, 1010, \cdots\} \subset \Sigma^*.$$

- C^* *Language*: the collection of well-formed C^* programs is the subset of all possible strings over the alphabet of C^* language.

Language over an Alphabet (*cont.*)

- C^* *program*:

```
int main() { printf("Hi\n") ;}
```

- *Not a* C^* *program*:

```
int main { printf("Hi\n") ;}
```

Σ^* is Denumerable

Let there are n symbols in the alphabet Σ. We may view elements of Σ as the basic symbols (excluding zero) of a *radix*-$(n+1)$ number system and Σ^* as a proper subset of natural numbers represented in the system.

- $\Sigma = \{a, b, c\}$: If we view this set as $\{1, 2, 3\}$, $\Sigma^* = \{\varepsilon, a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, \cdots\}$ may be viewed as $\{0, 1, 2, 3, 11, 12, 13, 21, 22, 23, 31, 32, 33, 111, \cdots\}$.
- There is a natural *bijection* from the transformed Σ^* to $\mathbb{N}$.

Σ^* is Denumerable (*cont.*)

- Injection : i_{Σ^*}

0	1	2	3	11	12	13	21	22	23	31	$\cdots$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0	1	2	3	5	6	7	9	10	11	13	$\cdots$

- This is an infinite set hence there is an *one-to-one* map from $\mathbb{N}$.
- Hence the *bijection* f_7 by *Schörader-Bernstein* theorem.

Infinite Language over Σ

Any *infinite* language L over Σ is *denumerable*.

- $i_L : L \longrightarrow \Sigma^*$, so that $i_L(x) = x$ is *one-to-one*.
- $i_{\Sigma^*} \circ i_L : L \longrightarrow \mathbb{N}$ is *one-to-one*.
- There is an *one-to-one* map from $\mathbb{N}$ to L, as L is infinite
- Therefore the *bijection* by *Schörader-Bernstein* theorem.

C^* Programs

Every C^* program is a string over the C^* alphabet. Hence there are *denumerably* many C^* programs.

Only Two Species?

Is it the case that there are only two types of sets - finite and denumerable?