Set, Relation and Function

Sets and Basic Operations on a Set

- *Member of a Set* : Let A be a set. *x is a member of A*, $x \in A$; *y is not a member of A*, $y \notin A$.
- *Empty Set* : $\{\}$ or $\emptyset$. Empty set does not have any member.
- *Non-empty Set* : $\{\{\}\}$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, $\{0, 2, 4, \cdots\}$.
- *Set union* : Let A and B be sets. *A union B* is a set, writen and defined as,

$$A \cup B = \{x \ : \ x \in A \text{ or } x \in B\}.$$

Sets and Operations (*cont.*)

- *Set intersection* : Let A and B be sets.
  A *intersection* B is a set, writen and defined as,

  $$A \cap B = \{x \ : \ x \in A \text{ and } x \in B\}.$$

- *Set Minus* : Let A and B be sets. A *minus* B is a set, writen and defined as,

  $$A \setminus B = \{x \ : \ x \in A \text{ and } x \notin B\}.$$

- *Disjoint Union* : Let A and B be sets.
  A *disjoint union* B is a set, writen and defined as,

  $$A \oplus B = (A \setminus B) \cup (B \setminus A)$$

Examples

Let $A = \{a, p, r, x, z\}$, $B = \{b, q, r, y, z\}$, and $C = \{b, c, q, t, y\}$ be sets.

- $A \cup B = \{a, p, r, x, z\} \cup \{b, q, r, y, z\} = \{a, b, p, q, r, x, y, z\}$.
- $A \cap B = \{a, p, r, x, z\} \cap \{b, q, r, y, z\} = \{r, z\}$.
- $A \cap C = \{a, p, r, x, z\} \cap \{b, c, q, t, y\} = \emptyset$.

A and C do not have any common element and the intersection is a *null set*.

Examples (*cont.*)

- $A \setminus B = \{a, p, r, x, z\} \setminus \{b, q, r, y, z\} = \{a, p, x\}$.
- $A \setminus C = \{a, p, r, x, z\} \setminus \{b, c, q, t, y\} = \{a, p, r, x, z\} = A$.
- $A \oplus B = (A \setminus B) \cup (B \setminus A) = \{a, p, x\} \cup \{b, q, y\} = \{a, b, p, q, x, y\}$.
- $A \oplus C = (A \setminus C) \cup (C \setminus A) = \{a, p, r, x, z\} \cup \{b, c, q, t, y\} = \{a, b, c, p, q, r, x, y, z\}$.

Sets and Operations (*cont.*)

- *Subset* : Let A and B be sets. A *is a subset of* B is writen and defined as,

$$A \subseteq B \equiv \text{ if } x \in A \text{ then } x \in B.$$

- *Proper subset* : Let A and B be sets.
  A *is a proper subset of* B is writen and defined as,

$$A \subset B \equiv A \subseteq B \text{ and } A \neq B$$

- *Equality of sets*: Two sets A and B are equal if they have the same elements i.e. $A \subseteq B$ and $B \subseteq A$.

Examples

Let $A = \{a, p, r, x, z\}$, $B = \{p, r, x\}$, and $C = \{b, q, r, y, z\}$ be sets.

- B is a *proper subset* of A as p, r and x are elements of A.
- A is a subset of A itself but not a proper subset.
- C is not a subset of A as $b \in C$, but $b \notin A$.

Examples (*cont.*)

- Let A be the set of even natural numbers.
- Let B be the set of natural numbers whose binary representation has a zero (0) in the *least significant bit*.
- $A = B$ as each element of A satisfies the property of B (i.e. $A \subseteq B$) and vice versa ($B \subseteq A$).

Sets and Operations (*cont.*)

- *Powerset* : Let A be a set. The *powerset* of A is a set. It is writen and defined as,

$$\mathbf{2}^A \text{ or } \mathcal{P}A = \{B : B \subseteq A\}.$$

A Few Properties of Set Operations

Let A and B be sets.

- *Identity Property* :

$$A \cup \emptyset = A = \emptyset \cup A.$$

- *Zero Property* :

$$A \cap \emptyset = \emptyset = \emptyset \cap A.$$

- *Commutative Property* :

$$A \cup B = B \cap A, \; A \cap B = B \cap A.$$

A Few Properties of Set Operations (*cont.*)

Let A, B, and C be sets.

- *Associative Property* :

  $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$.

- *Distributive Property* :

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C),$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

  Also the dual of it.

Proof

Let A, B, and C be sets. Any proof of these properties depends on the properties of **not**, **or** and **and**. We try to prove that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

Proof (*cont.*)

$$
\begin{array}{rcl}
x \in (A \cup B) \cap C & \textit{iff} & x \in (A \cup B) \text{ and } x \in C, \\
& \textit{iff} & (x \in A \text{ or } x \in B) \text{ and } x \in C, \\
& \textit{iff} & (x \in A \text{ and } x \in C) \text{ or } \\
& & (x \in B \text{ and } x \in C) \\
& \textit{iff} & (x \in A \cap C) \text{ or } (x \in B \cap C) \\
& \textit{iff} & x \in (A \cap C) \cup (B \cap C) \\
(A \cup B) \cap C & = & (A \cap C) \cup (B \cap C)
\end{array}
$$

Union and Intersection over a Family

Let $\mathcal{A} = \{A_i\}_{i \in I}$ be a collection of sets so that $A_i \subseteq B$, for all $i \in I$.

- *Union* : union of all the elements of $\mathcal{A}$,

$$\bigcup \mathcal{A} \text{ or } \bigcup_{i \in I} \mathcal{A} = \{x \in B : x \in A_i, \text{ for some } i \in I\}.$$

- *Intersection* : intersection of all the elements of $\mathcal{A}$,

$$\bigcap \mathcal{A} \text{ or } \bigcap_{i \in I} \mathcal{A} = \{x \in B : x \in A_i, \text{ for all } i \in I\}.$$

Examples

Let $\mathcal{A} = \{\{a, c, p, x, z\}, \{b, c, q, x, y\}, \{a, c, x, y, z\}\}$.

$$\begin{aligned}
\bigcup \mathcal{A} &= \bigcup \{\{a, c, p, x, z\}, \{b, c, q, x, y\}, \{a, c, x, y, z\} \\
&= \{a, c, p, x, z\} \cup \{b, c, q, x, y\} \cup \{a, c, x, y, z\} \\
&= \{a, b, c, p.q, a, y, z\}
\end{aligned}$$

$$\begin{aligned}
\bigcap \mathcal{A} &= \bigcap \{\{a, c, p, x, z\}, \{b, c, q, x, y\}, \{a, c, x, y, z\} \\
&= \{a, c, p, x, z\} \cap \{b, c, q, x, y\} \cap \{a, c, x, y, z\} \\
&= \{c, x\}
\end{aligned}$$

Do You Agree?

- Let $\mathcal{A} = \{A_i\}_{i \in I}$ be a collection of subsets of the set B. The set $\bigcup \mathcal{A}$ is defined to be the collection of x's satisfying the following property.

$$(x \in B \text{ and } x \in A_i \text{ for some } i \in I)$$

If the index set $I = \emptyset$ (empty), the statement '$x \in A_i$ for some $i \in I$' is not *true* for any $x \in B$ and hence $\bigcup \mathcal{A} = \emptyset$.

Do You Agree? (*cont.*)

- The set $\bigcap \mathcal{A}$ is defined to be the collection of x's satisfying the following property.

$$(x \in B \text{ and } x \in A_i \text{ for every } i \in I)$$

If the index set I is empty, the statement '$x \in A_i$ for every $i \in I$' is *true* for any $x \in B$, because there is no A_i present in $\mathcal{A}$ to make it *false* (*vacuously true*). Hence $\bigcap \mathcal{A} = B$!

Ordered Pair and Cartesian Product

Let A and B be sets.

- An ordered pair from A to B is (a, b), where $a \in A$ and $b \in B$.
- Two ordered pairs (a, b) and (c, d) are equal if $a = c$ and $b = d$.
- *Cartesian Poduct* : the *cartesian*[a] *product* of A and B is a set, writen and defined as

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

[a]French mathematician and philosopher, Rene Decartes (1596-1650).

Example

Let $A = \{x, y\}$ and $B = \{p, q, x\}$ be sets.

- $A \times B = \{(x, p), (x, q), (x, x), (y, p), (y, q), (y, x)\}$.
- $A \times A = \{(x, x), (x, y), (y, x), (y, y)\}$.
- If either of A or B is an empty set, the statement '$x \in A$ and $p \in B$' is always *false*. Hence

$$A \times \emptyset = \emptyset = \emptyset \times A$$

Binary Relation

Let A and B be sets.

- *Binary relation* : a binary relation R from A to B is a subset of $A \times B$. A is the *domain* of the relation R and B is the *codomain* of it.
- A *binary reation* R on A is a subset of $A \times A$.
- If $(a, b) \in R$, a relation from A to B, we write aRb i.e. 'a is R related to b'.

Examples

Let $A = \{x, y\}$ and $B = \{p, q, x\}$ be sets.

- $R = \{(x, p), (x, q)\}$ is a biary relation from A to B. We may write xRp, xRq .
- $S = \{(p, p), (q, q), (x, x)\}$ is a binary relation on B. We may write pSp, qSq, and xSx
- $< \; = \{(0, 1), (0, 2), (0, 3), \cdots, (1, 2), (1, 3), \cdots\}$ is a binary relation on $\mathbb{N}$. We write $3 < 10$.

A Function or a Map

Let A and B be sets.

- *Function/Map* : a function or a map f from A to B is an assignment of exactly one element of B to each element of A. If $b \in B$ is assigned to $a \in A$ by the function f, we write $f(a) = b$, or $fa = b$, or $a \overset{f}{\mapsto} b$. The function is written as $f : A \longrightarrow B$ or $A \overset{f}{\longrightarrow} B$. A is the *domain* and B is the *codomain* of the function.
- A *function* f on A is an assignment of exactly one element of A itself to each element of A. Here both *domain* and *codomain* are the same A.

Argument and Value

Let f be a function from a set A to a set B. An element $a \in A$ is called an *argument* of the function and $f(a)$ is the corresponding *value* assigned to 'a' by the function. $f(a)$ is called the *image* of $a \in A$ and a is the *coimage* of $f(a)$.

Example

Let $A = \{x, y, z\}$ and $B = \{p, q, x, z\}$ be sets.

- Function $f : A \longrightarrow B$,

$$x \mapsto q,\ y \mapsto q,\ z \mapsto p$$

- Function $g : A \longrightarrow A$,

$$x \mapsto y,\ y \mapsto x,\ z \mapsto z$$

- Function $! : \mathbb{N} \longrightarrow \mathbb{N}$,

$$0 \mapsto 1,\ 1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 6,\ 4 \mapsto 24, \cdots$$

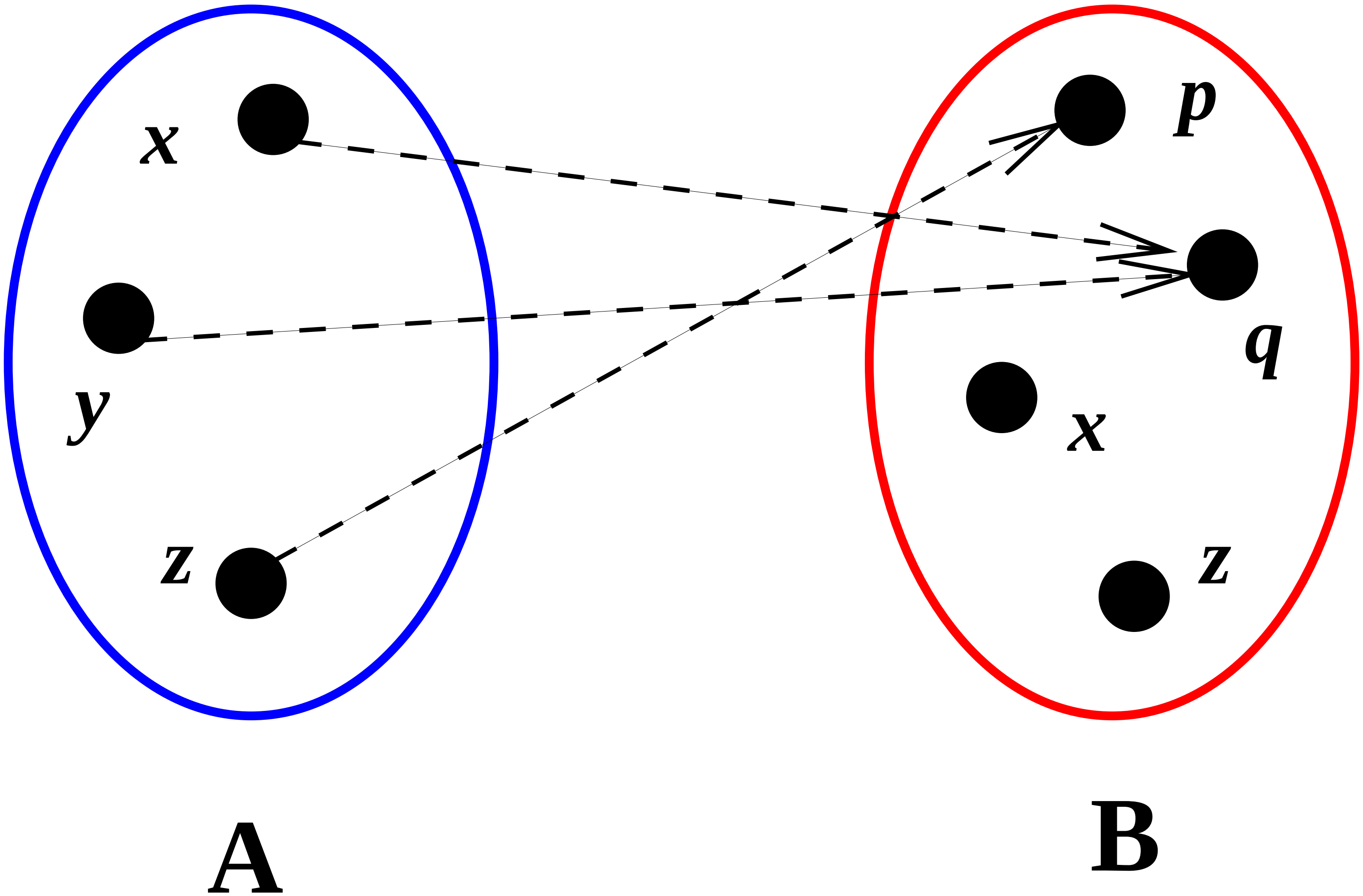

Figure 1: **Function**

Graph of a Function

Let $f : A \longrightarrow B$ be a function, the *graph* of f is the following binary relation from A to B.

$$graph(f) = \{(a, f(a)) \ : \ a \in A\}$$

In *set theory* the *graph* of a function is called the function or map (*extensionally*).

Examples

$$graph(f) = \{(x, q), (y, q), (z, p)\}$$

$$graph(g) = \{(x, y), (y, x), (z, z)\}$$

$$graph(!) = \{(0, 1), (1, 1), (2, 2), (3, 6), (4, 24), (5, 120), \cdots\}$$

A Bit of Logic : Conditional Statement

A *conditional statement* is written in any one of the following form.

- If A then B
- A only if B.
- A imples B ($A \Rightarrow B$, or $A \supset B$).
- if not B then not A.

In the first three cases, A is the *antecedent* and B is the *consequence*. The last statement is in contrapositive form.

Value of a Conditional Statement

A	B	$A \Rightarrow B$
false	*false*	*true*
false	*true*	*true*
true	*false*	*false*
true	*true*	*true*

Material Implcation

- If the *antecedent* of a conditional statement is *false*, the conditional statement is *true* irrespective of the *consequence*, *vacuously true*.
- It is *false* only when the *antecedent* is *true* but the *consequence* is *false*.
- This conditional statement or implecation is called **material implecation**.

Use of Material Implecation

- Let A and B be sets.

$$A \subseteq B \equiv (x \in A) \Rightarrow (x \in B), \text{for all } x.$$

If $A = \emptyset$, then $x \notin A = \emptyset$, for all $x \in B$. This makes the statement $(x \in \emptyset) \Rightarrow (x \in B)$, *true*. Hence $\emptyset \subseteq \mathbf{B}$, for any set B.

- As a special case $\emptyset \subseteq \emptyset$ and this is the only subset of a *null set*.
- $\mathbf{2}^{\emptyset}$ or $\mathcal{P}\emptyset = \{\emptyset\}$ or $\{\{\}\}$, a set with only one element, the *null set*.

Examples

Let $A = \{a, b, c\}$

- The subsets of A are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$.
- $\mathbf{2}^A$ or $\mathcal{P}A =$ $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Another Use

- Let A and B be sets. If exactly one element $b \in B$ is assigned to each element $a \in A$, we say that there is a function from A to B.

  If the set $A = \emptyset$, there is no $a \in A$, and the *antecedent* is *vacuously* true. Hence there is a unique function ϕ_B from the *null set* to any set B. The graph of this function is also an empty set, $graph(\phi_B) = \emptyset$.

Biconditional Statement : *iff*

An *if and only if* is written in any one of the following form.

- *A if and only if B*.
- *A iff B*.
- $A \Leftrightarrow B$, or $A \equiv B$.

This statement is equivalent to a conjunction of two implecations.

- *A if B* (if B then A or $B \Rightarrow A$) and *A only if B* ($A \Rightarrow B$).

Equality of Two Functions

Two functions $f, g : A \longrightarrow B$ are said to be equal if for every argument the values are equal i.e.

$$f = g \textit{ iff } f(a) = g(a), \text{ for all } a \in A.$$

This is called equality by *extensionality*. In other words $graph(f) = graph(g)$.

Function Composition

Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$. Given f and g, we can define a new function from A to C, *the composition* of g with f, $g \circ f : A \longrightarrow C$, such that,

$$(g \circ f)(a) = g(f(a)) \in C, \text{ for all } a \in A.;$$

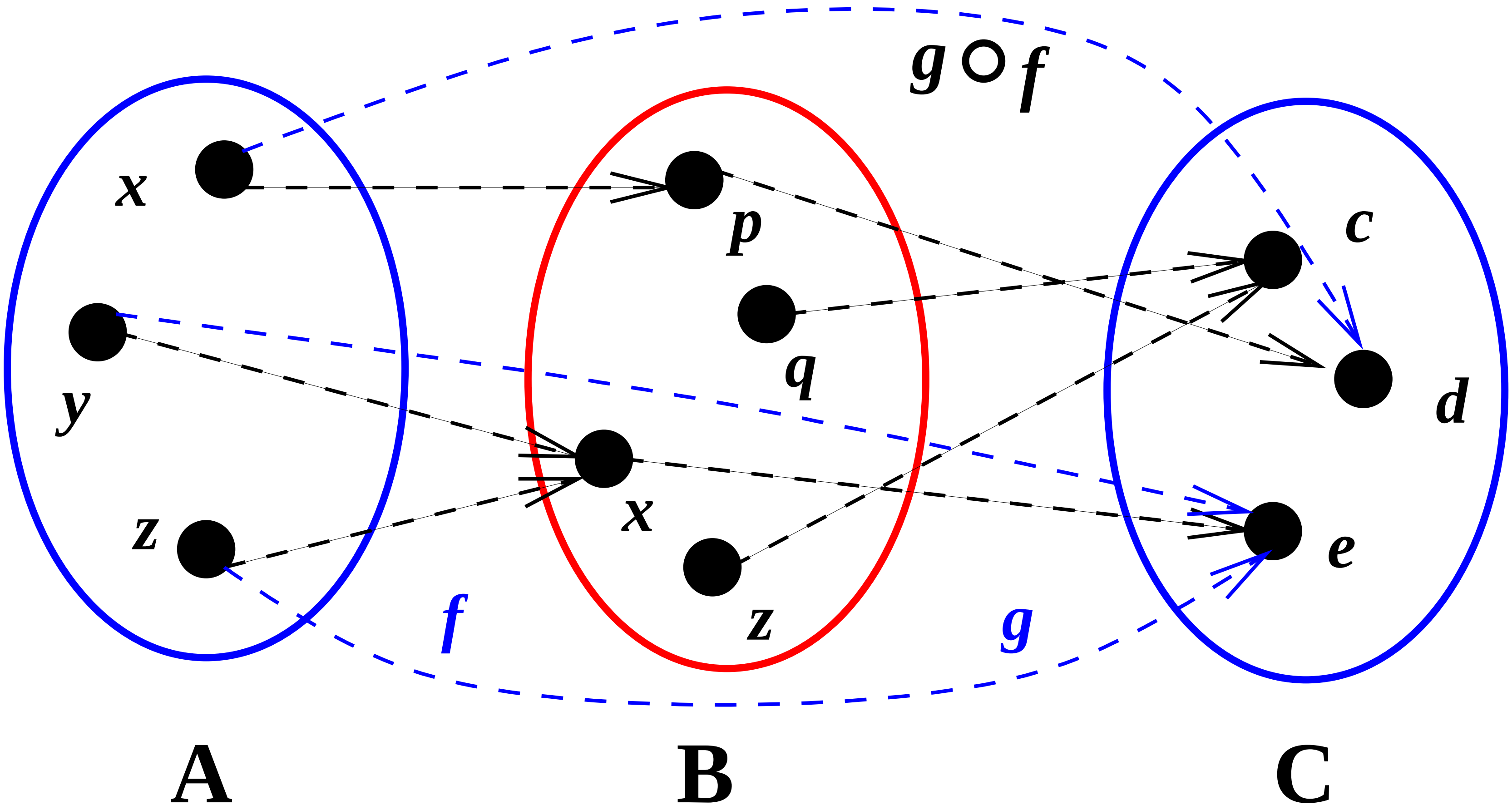

Figure 2: **Function Composition**

Identity Function

For every set A, there is a function $1_A : A \longrightarrow A$, called the *identity function* such that

$$1_A(a) = a, \text{ for all } a \in A.$$

Properties of Function Composition

Let $f : A \longrightarrow B$, $g : B \longrightarrow C$ and $h : C \longrightarrow D$.

- $(h \circ (g \circ f)) = ((h \circ g) \circ f)$: function composition is *associative* like ordinary addition and multiplication.
- $1_B \circ f = f = f \circ 1_A$: the identity function behaves like '0' in addition and '1' in multiplication.
- If $A = B$, $1_A = 1_B$ and $1_A \circ f = f = f \circ 1_A$.
- Function composition in general is not *commutative* i.e. $g \circ f \neq f \circ g$ (not defined in this case).