Formal Language and Automata Theory (CS21004)

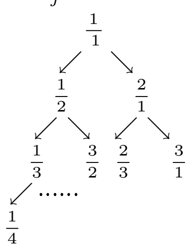
Tutorial - II

Class: CSE 2^{nd} Year *Date*: 12^{th} January, 2010

Exercise 1. Give a bijection from $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, 3, 4, \cdots\cdots\}$.

Exercise 2. Neil Calkin and Herbert Wilf[1], following the work of Moritz Abraham Stern and Achille Brocot[2], have given an enumeration without duplication of rationals in their lowest terms. Calkin-Wilf's construction is an infinite binary tree where each node is labelled by a rational $\frac{i}{j}$ so that the $gcd(i, j) = 1$.

The root of the tree is labelled by $\frac{1}{1}$, if a node is labelled by $\frac{i}{j}$, then its left-child is labelled by $\frac{i}{i+j}$ and the right-child is labelled by $\frac{i+j}{j}$.



Prove the following facts about the construction.

i. Every fraction $\frac{i}{j}$ in is in its lowest terms.

ii. Every fraction $\frac{i}{j} > 0$ and in its lowest terms, will appear in the tree.

iii. No reduced fraction $\frac{i}{j}$ can appear more than once in the tree.

iv. The denominator of the n^{th} fraction (in the breadth-first order) is same as the numerator of the $(n+1)^{\text{th}}$ fraction. We call $\frac{1}{1}$ as the 0^{th} fraction.

v. The numerators of the Calkin-Wilf's sequence of fractions give the following sequence of integers:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \cdots\cdots \implies 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, \cdots\cdots$$

Give a recurrence relation for the sequence, i.e. $f(0) = 1$, $f(2n+1) = ?$ (left-child) and $f(2n+2) = ?$, for all $n \geqslant 0$.

vi. How do you construct the n^{th} fraction from the sequence?

Exercise 3. Consider the language $L_1 = \{\varepsilon, a, aa, aaa, \cdots\cdots\}$. We can give an inductive definition of the set L_1 as follows:

i. *Basis*: $\varepsilon \in L_1$,

ii. *Induction*: If $x \in L_1$, then so is ax,

iii. Nothing else is in L_1. In other words, L_1 is the smallest set satisfying (i) and (ii).

Take another example, $L_2 = \{ab, aabb, aaabbb, \cdots\cdots\}$, we define L_2 as follows:

i. *Basis*: $ab \in L_1$,

1. Recounting the Rationals, N. Calkin & H. Wilf, American Mathematical Monthly, 107, (2000), 360-363. Department of Mathematics, University of Pennsylvania.

2. 1858, 1861

ii. *Induction*: If $x \in L_1$, then so is axb,

iii. Nothing else is in L_1.

Give similar definitions for the following languages.

a) $L_3 = \{x \in \{a, b\}^\star : |x|_a \text{ is even}\}$.

b) $L_4 = \{x \in \{a, b\}^\star : |x|_a = |x|_b\}$.

c) $L_5 = \{x \in \{a, b\}^\star : |x|_a \text{ is odd and } |x|_b \text{ is even }\}$.

Exercise 4. Prove or disprove the following results:

i. $(L_1 L_2)^R = L_2^R L_1^R$.

ii. $(L^\star)^R = (L^R)^\star$.

iii. $(L_1 \backslash L_2)^R = L_2^R / L_1^R$, reverse of the *right quotient* of L_1 with respect to L_2 is equal to the *left quotient* of the reverse of L_1 with respect to the reverse of L_2.

Exercise 5. Prove that the following grammar describes the language $L_5 = \{x \in \{a, b\}^\star : |x|_a = |x|_b\}$:

$G = (\{S\}, \{a, b\}, P, S)$, where P: $S \rightarrow \varepsilon$, $S \rightarrow aSb$, $S \rightarrow bSa$, $S \rightarrow SS$.