Computer Science & Engineering Department

IIT Kharagpur

Computational Number Theory: CS60094

Tutorial I

Instructor: Goutam Biswas *Spring Semester 2014-2015*

1. Give the Euclid's proof that the set of primes $\mathbb{P}$ is infinite.

2. Prove that any two element of the Fermat sequence, $F_n = 2^{2^n} + 1$, $n \in \mathbb{N}$, are relatively prime. What is your conclusion from this.

3. Assume that the collection of all primes $\mathbb{P}$ is finite, and let p be the largest prime. Prove that any prime divisor q of $2^p - 1$ is larger than p. What is your conclusion?

4. If $2^n + 1$ is a prime, then prove that n is a power of 2.

5. If there are finite number of primes, then the sum $\sum_{p\in\mathbb{P}} \frac{1}{p}$ converges. If we can prove that the sum diverges, then certainly there are infinite number of primes. We assume that $\sum_{p\in\mathbb{P}} \frac{1}{p}$ converges. Let $p_1 = 2, p_2 = 3, p_3 = \cdots$ be the sequence of primes in increasing order. There is some k such that $\sum_{i\geq k+1} \frac{1}{p_i} < \frac{1}{2}$ (as the sequence converges). We call $\mathbb{P}_s = \{p_1, \cdots, p_k\}$, the set of *small primes* and $\mathbb{P}_l = \{p_{k+1}, \cdots\}$, the set of *large primes*.
   Let N be any positive integer. We partition $\{1, \cdots, N\} = A_s \cup A_l$, where all prime factors of each element of A_s are *small primes* and each element of A_l have at least one *big prime* factor. We include 1 in A_s.
   It is clear that $N = N_s + N_l$, where $N_s = |A_s|$ and $N_l = |A_l|$.

   (a) Prove that $N_l < \frac{N}{2}$.

   (b) Prove that $N_s \leq 2^k\sqrt{N}$.

   (c) Prove that for some large N, $N > N_s + N_l$. What is your final conclusion?

6. Prove that the n^{th} prime $p_n \leq 2^{2^{n-1}}$.

7. Let $\pi(x) = |\{p \in \mathbb{P} : \ p \leq x\}|$, where $x \in \mathbb{R}^+$ is a positive real, and $n \leq x < n + 1$, where $n \in \mathbb{N}$. Prove the following facts:

(a) Prove that $\log x \leq \sum_{i=1}^{n} \frac{1}{i}$. Also prove that $\log x \leq \sum \frac{1}{i}$, the sum is over all $i \in \mathbb{N}$ whose prime factors are in $\pi(x)$.

(b) Prove that $\log x \leq \pi(x) + 1$.

(c) What is your final conclusion?