Relations
--- Binary Relations

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What is a relation?

- The mathematical concept of relation is based on the common notion of relationships among objects:
  - One box is heavier than the other
  - One man is richer than the other
  - An event occurs prior to the other

Ordered n-tuple

- For n>0, an ordered n-tuple (or simply n-tuple) with ith component a_i is a sequence of n objects denoted by $<a_1,a_2,\ldots,a_n>$. Two ordered n-tuples are equal iff their ith components are equal for all i, $1<=i<=n$.
- For n=2, ordered pair
- For n=3, ordered triple

Cartesian Product

- Let $\{A_1,A_2,\ldots,A_n\}$ be an indexed collection of sets with indices from 1 to n, where n>0. The cartesian product, or cross product of the sets A_1 through A_n, denoted by $A_1 \times A_2 \times \ldots \times A_n$, or $\times_{i=1}^{n} A_i$ is the set of n-tuples $\langle a_1,a_2,\ldots,a_n \rangle | a_i \in A_i\}$. When $A_i=A$, for all i, then $\times_{i=1}^{n} A_i$ will be denoted by A^n.

Examples

- Let $A=\{1,2\}$, $B=\{m,n\}$, $C=\{0\}$, $D=\Phi$.
  - $AxB=\{<1,m>,<1,n>,<2,m>,<2,n>\}$
  - $AxC=\{<1,0>,<2,0>\}$
  - $AxD= \Phi$
- When A and B are real numbers, then AxB can be represented as a set of points in the Cartesian Plane. Let, $A=\{x|1\leq x\leq 2\}$ and $B=\{y|0\leq y\leq 1\}$. Then
  - $AxB=\{<x,y>|\ 1\leq x\leq 2\ \Lambda\ 0\leq y\leq 1\}$

Theorems

1. $A X (B \cup C) = (A X B) \cup (A X C)$
2. $A X (B \cap C) = (A X B) \cap (A X C)$
3. $(A \cup B) X C = (A X C) \cup (B X C)$
4. $(A \cap B) X C = (A X C) \cap (B X C)$

Proof of 1

$$< x, y > \in A \times (B \cup C) \Leftrightarrow x \in A \wedge y \in (B \cup C)$$

$$\Leftrightarrow x \in A \wedge (y \in B \vee y \in C)$$

$$\Leftrightarrow (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$$

$$\Leftrightarrow (< x, y > \in A \times B) \vee (< x, y > \in A \times C)$$

$$\Leftrightarrow < x, y > \in (A \times B) \cup (A \times C)$$

The rest of the proofs are similar.

What is a relation mathematically?

- Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation R on $\times_{i=1}^{n} A_i$ is a subset of $\times_{i=1}^{n} A_i$. If R=Φ, then R is called the empty or void relation. If $R = \times_{i=1}^{n} A_i$ then R is the universal relation. If $A_i = A$ for all i, then R is called an ***n-ary relation on A***.
- If n=1, unary
- If n=2, binary
- Ternary…

Number of n-ary relations

- If A_i has r_i elements, then $\times_{i=1}^{n} A_i$ has $\prod_{i=1}^{n} r_i$ elements
- The number of n-ary relations is the cardinal number of the power set of the cartesian product of the A_is.
- Thus, the number of relations is

$$\boxed{2^{\prod_{i=1}^{n} r_i}}$$

Equality of relations

- Let R_1 be an n-ary relation on $\times_{i=1}^{n} A_i$ and R_2 be an m-ary relation on $\times_{i=1}^{m} B_i$. Then $R_1=R_2$ iff n=m, and $A_i=B_i$ for all i, $1 \leq i \leq n$, and $R_1=R_2$ are equal sets of ordered n-tuples.
- Every n-ary relation on a set A, corresponds to an n-ary predicate with A as the universe of discourse.
- A unary relation on a set A is simply a subset of set A.

Binary Relations

- They are frequently used in abstraction in CS
- Various data structures, like trees and graphs can be modeled as binary relations and vice versa.
- We shall see techniques and methods to analyze.

Binary Relations

- Let A, B be any two sets.
- A *binary relation* R from A to B, written (with signature) $R{:}A\leftrightarrow B$, is a **subset of $A\times B$**.
  - *E.g.*, let < : $\mathbf{N}\leftrightarrow\mathbf{N}$:≡ $\{<n,m> \mid n < m\}$
- The notation $a\ R\ b$ or aRb means $<a,b>\in R$.
  - *E.g.*, $a < b$ means $(a,b)\in <$
- If aRb we may say "a is related to b (by relation R)", or "a relates to b (under relation R)".
- A binary relation R corresponds to a predicate function $P_R{:}A\times B\rightarrow\{\mathbf{T},\mathbf{F}\}$ defined over the 2 sets A,B; *e.g.*, "eats" :≡ $\{<a,b> \mid$ organism a eats food $b\}$

Domain and Co-domain

- Let R be a binary relation over AxB.
- Domain: Set A
- Co-domain: Set B
- $<a,b> \in R => aRb$
- $<a,b> \notin R => a \not{R} b$

Complementary Relations

- Let R:$A \leftrightarrow B$ be any binary relation.
- Then, $\cancel{R}$:$A \leftrightarrow B$, the *complement* of R, is the binary relation defined by
  $$\cancel{R} :\equiv \{<a,b> \mid (a,b) \notin R\}$$

Example: $< \; = \{(a,b) \mid (a,b) \notin <\} = \{(a,b) \mid \neg a < b\} = \underline{\geq}$

Inverse Relations

- Any binary relation $R{:}A{\leftrightarrow}B$ has an *inverse* relation $R^{-1}{:}B{\leftrightarrow}A$, defined by

  $$R^{-1} :\equiv \{(b,a) \mid (a,b)\in R\}.$$

  E.g., $\mathbf{<}^{-1} = \{(b,a) \mid a<b\} = \{(b,a) \mid b>a\} = \mathbf{>}$.
- *E.g.*, if R:People→Foods is defined by
  $aRb \Leftrightarrow$ *a eats b*, then:
  $b\, R^{-1}\, a \Leftrightarrow$ *b is eaten by a*. (Passive voice.)

Relations on a Set

- A (binary) relation from a set A to itself is called a relation *on* the set A.
- *E.g.*, the "<" relation from earlier was defined as a relation *on* the set **N** of natural numbers.
- The *identity relation* $\mathbf{I}_A$ on a set A is the set $\{(a,a)|a\in A\}$.

Representing Relations

– With a zero-one matrix.
– With a directed graph.

Using Zero-One Matrices

- To represent a relation R by a matrix $\mathbf{M}_R = [m_{ij}]$, let $m_{ij} = 1$ if $(a_i,b_j) \in R$, else 0.
- *E.g.*,A={1,2,3}, B={1,2}. Let R be the relation from A to B containing (a,b) s.t a is in A and b is in B and a>b.
- The 0-1 matrix representation

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

When A=B, we have a square matrix

So, what is complement of R?

- A={1,2,3}, B={1,2}. Let R be the relation from A to B containing (a,b) s.t a is in A and b is in B and a>b
- Complement of R = {(a,b)|not(a>b)}

  ={(a,b)|a<=b}
- 0-1 matrix is:

$$M_{\overline{R}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

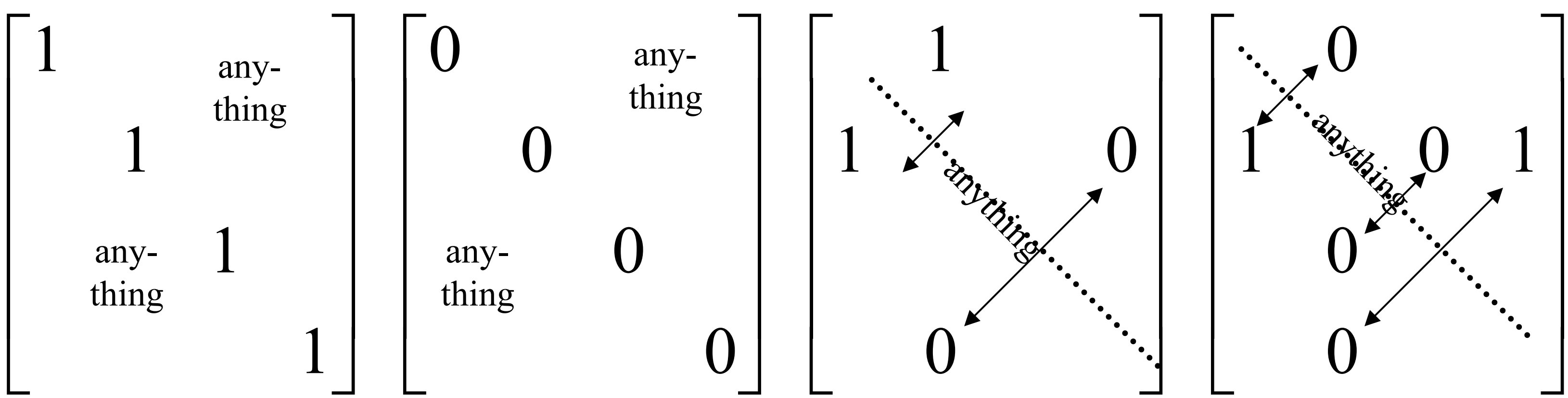
We can obtain by the element wise bit complement of the matrix.

Types of Relations

- Let R be a binary relation on A:
  - R is reflexive if xRx for every x in A
  - R is irreflexive if $x \not R x$ for every x in A
  - R is symmetric if xRy implies yRx for every x,y in A
  - R is antisymmetric if xRy and yRx together imply x=y for every x,y in A
  - R is transitive if xRy and yRz imply xRz for every x,y,z in A

Zero-One Reflexive, Symmetric

– These relation characteristics are very easy to recognize by inspection of the zero-one matrix.



Reflexive: all 1's on diagonal	*Irreflexive*: all 0's on diagonal	*Symmetric*: all identical across diagonal	*Antisymmetric*: all 1's are across from 0's

Tell what type of relation

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{(Reflexive, Symmetric)}$$

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{(Neither Reflexive nor irreflexive, Symmetric)}$$

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(Reflexive, Anti-Symmetric)}$$

Operations on 0-1 Matrix

- Union and Intersection of relations can be obtained by join and meet of the Binary matrices

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$
$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

Operations on 0-1 Matrix

$$M_{R_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{R_1 \cup R_2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad M_{R_1 \cap R_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Composition of relations

- R: A→B, S:B→C

$$\boxed{S \circ R : A \to C}$$

- Suppose, A, B and C have m, n and p elements
- M_S:[s_{ij}] (nxp), M_R:[r_{ij}](mxn), $M_{S.R}$:[t_{ij}](mxp)
- (a_i,c_j) belongs to S.R iff there is (a_i,b_k) belonging to R and (b_k,c_j) belonging to S for some k.
- Thus t_{ij}=1 iff r_{ik}=1 and s_{kj}=1, for some k.
- Thus, $\boxed{M_{S \circ R} = M_R \odot M_S}$

Example of composition

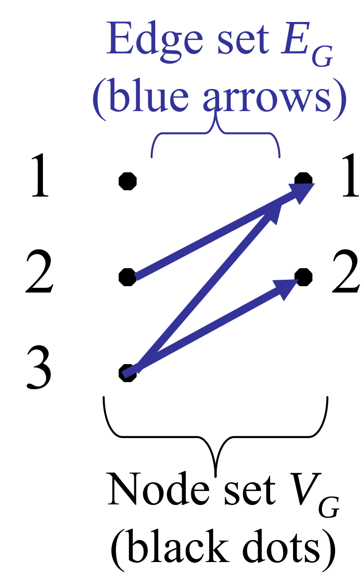
$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathrm{M}_{\mathrm{S} \circ \mathrm{R}} = \mathrm{M}_{\mathrm{R}} \odot \mathrm{M}_{\mathrm{S}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Using Directed Graphs

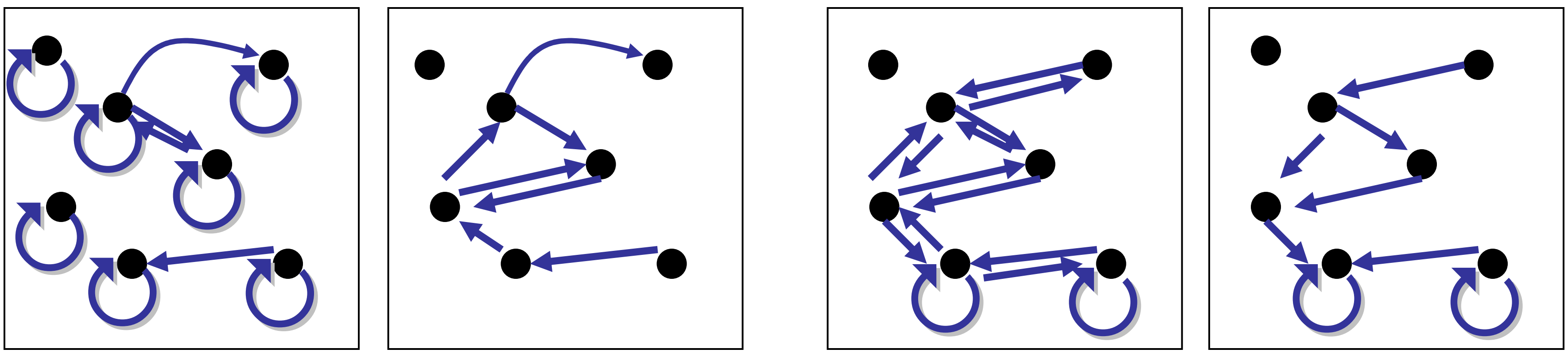
- A *directed graph* or *digraph* $G=(V_G,E_G)$ is a set V_G of *vertices (nodes)* with a set $E_G \subseteq V_G \times V_G$ of *edges (arcs,links)*. Visually represented using dots for nodes, and arrows for edges. Notice that a relation $R{:}A \leftrightarrow B$ can be represented as a graph $G_R=(V_G=A\cup B, E_G=R)$.

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$



Digraph Reflexive, Symmetric

It is extremely easy to recognize the reflexive/irreflexive/ symmetric/antisymmetric properties by graph inspection.



Reflexive: Every node has a self-loop

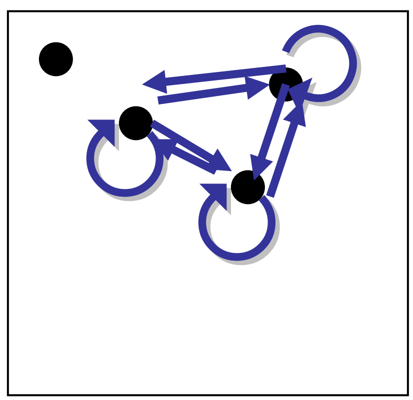
Irreflexive: No node links to itself

Symmetric: Every link is bidirectional

Antisymmetric: No link is bidirectional

A Question discussed in class

- Does symmetricity and transitivity imply reflexivity ?
- Reason of doubt:
  - aRb=>bRa (symmetricity)
  - This implies aRa (transitivity)
  - So, R is reflexive!
- Clarification:



Symmetric, Transitive
But not
Reflexive..

Closure of Relations

Closure?

- Let R be a relation on a set A
- R may or may not have a property P
- Define S, as the relation which has the property P AND
- S contains R AND
- S is the subset of every relation with property P and which contains R
- S is called the closure of R w.r.t P
- Closure may not exist.

Reflexive Closure

- R={(1,1),(1,2),(2,1),(3,2)} on the set A={1,2,3}
- Is R reflexive?
- How can we create an S (which is as small as possible) containing R which is reflexive?
- Add (2,2) and (3,3).
- S is reflexive and contains R
- Since, any reflexive relation on A must contain (2,2) and (3,3), all such relations must be a superset of S
- S is hence the reflexive closure.

Generalization

- Define $\Delta=\{(a,a)|a \in A\}$ **(Diagonal Relation)**
- $S=R \cup \Delta$
- S is the reflexive closure of R.

Symmetric Closure

- R={(1,1),(1,2),(2,2),(2,3),(3,1),(3,2)} on the set A={1,2,3}
- Is R symmetric?
- How can we create an S (which is as small as possible) containing R which is symmetric?
- Add (2,1) and (1,3).
- S is symmetric and contains R
- Since, any symmetric relation on A must contain (2,1) and (1,3), all such relations must be a superset of S
- S is hence the symmetric closure.

Generalization

- Define $R^{-1}=\{(b,a)|(a,b) \in R\}$
- $R=\{(1,1),(1,2),(2,2),(2,3),(3,1),(3,2)\}$
- $R^{-1}=\{(1,1),(2,1),(2,2),(3,2),(1,3),(2,3)\}$
- $S=R \cup R^{-1}$
  $=\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2)\}$
- S contains R
- All such relations contain S
- Thus, S is the symmetric closure.

Transitive Closure?

- R={(1,3),(1,4),(2,1),(3,2)}
- R is not transitive.
- So, add (1,2),(2,3),(2,4),(3,1).
- Does it become transitive?
- No, because say (3,2) and (2,4) are members but not (3,4).
- So, transitive closure is not that easy.

Composition of R with itself : R^n

- Let R be a relation on set A
- aRb => {(a,b) | (a,b)ЄR}
- Let R be a relation on the set A. The powers R^n, n=1, 2, 3, … are defined recursively by:

  $R^1=R$ and $R^{n+1}=R^n.R$

- Example: R={(1,1),(2,1),(3,2),(4,3)}

  R^2={(1,1),(2,1),(3,1),(4,2)}

Composition in DAG

- A path from a to b in DAG G, is a sequence of edges $(a,x_1),(x_1,x_2),\ldots,(x_{n-1},b)$. The path has length n. A path of length $n \geq 1$ that begins and ends at the same vertex is called a circuit or cycle.
- <u>Theorem:</u> *Let R be a relation on a set A. There is a path of length n, where n is a positive integer from a to b, iff (a,b) belongs to R^n.*

Proof

- **Base:** There is a path from a to b of length 1, iff (a,b) is in R.
- **Induction**: Assume theorem is true for n
- There is a path of length (n+1) between a and b, iff there is a path of length 1 between (a,c) and there is a path of length of n between (c,b) for some c.
- Hence, there is such a path iff $(a,c) \in R$ and $(c,b) \in R^n$ (inductive hypothesis)
- But there is such an element c iff $(a,b) \in R^{n+1}$

Theorem

- The relation R on a set A is transitive iff

$$\boxed{R^n \subseteq R}$$

- **If part:** If R^2 is a subset of R (special case) then R is transitive
- **Else part:**
  - Trivial proof for n=1
  - Assume if R is transitive R^n is a subset of R.
  - Consider (a,b)ЄR^{n+1}. Thus, there is an element c st (a,c) Є R and (c,b) Є R^n. By hypothesis, (c,b) Є R.
  - But R is transitive, so (a,c) Є R and (c,b) Є R means (a,b) Є R

Now lets look at the Problem of Transitive Closure

- Define, the connectivity relation consisting of the pairs (a,b) such that there is a path of length at least one from a to b in R.

$$\boxed{R^+ = \bigcup_{n=1}^{\infty} R^n}$$

- **Theorem:** The connectivity relation is the transitive closure
- Proof:
  - R^+ contains R
  - R^+ is transitive

To show that R^+ is the smallest!

- Assume a transitive S containing R
- R^+ is a subset of S^+ (as all paths in R are also paths in S)
- Thus, we have

$$\boxed{R^+ \subseteq S^+ \subseteq S \text{ (as S is transitive we have } S^n \subseteq S\text{)}}$$

Lemma

- ***Let, A be a set with n elements, and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n.***
- *Thus, the transitive closure is*

$$t(R)=\bigcup_{i=1}^{n} R^i$$

- *Proof follows from the fact* R^k *is a subset of t(R)*

Example

- Find the zero-one matrix of the transitive closure of

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R^+} = M_R \vee M_R^{[2]} \vee M_R^{[3]}$$

$$M_R^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, M_R^3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_{R^+} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Algorithm-1

- Procedure transitive-closure(M_R)

  $A=M_R$, $B=A$

  for i=2 to n

  begin

  $\quad A = A \odot B$

  $\quad B = B \vee A$

  end

  B is the answer

Complexity:
$n^2(2n-1)(n-1)+n^2(n-1)=$**$O(n^4)$**

Algorithm-2 (Roy-Warshall algorithm)

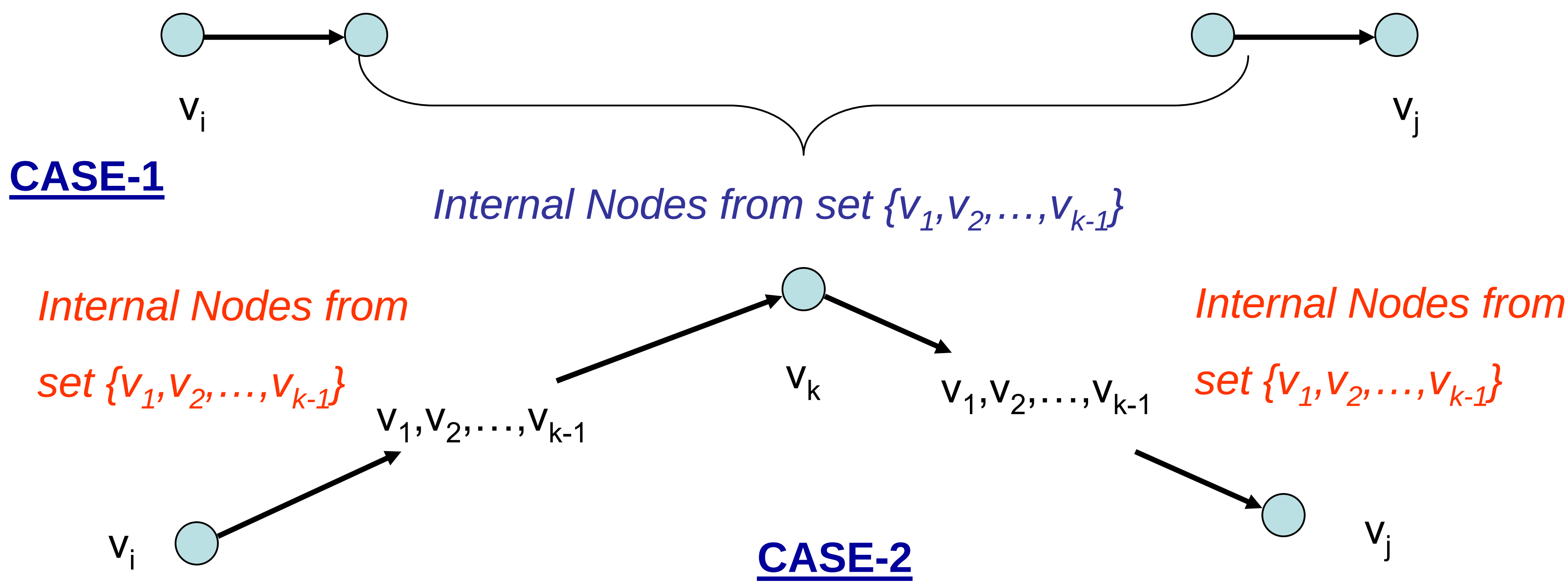
- Based on the construction of 0-1 matrices, $W_0, W_1, \ldots, W_n$, where $W_0 = M_R$ (0-1 matrix of the relation).
- Uses the concept of internal vertices of a path: If there is a path (a,b), namely, $(a, x_1, x_2, \ldots x_{m-1}, b)$
- Internal vertices: $x_1, x_2, \ldots x_{m-1}$
- The start vertex is not an internal vertex unless it is visited again, except as a last vertex
- The end vertex is not an internal vertex unless it has been visited before, except as a first vertex

So, what is the trick?

- Construct, $W_k=[w_{ij}^{(k)}]$, where $w_{ij}^{(k)}=1$, if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1,v_2,\ldots,v_k\}$, and 0 otherwise.
- $W_n=M_R^*$. Can you see why?
- But construction of W_n is easy than the boolean product of matrices.

Construct W_k

- $w_{ij}^{(k)}=1$, *if there is a path from* v_i *to* v_j *such that all the interior vertices of this path are in the set* $\{v_1,v_2,\ldots,v_k\}$*, and 0 otherwise.*



Computing W_k

- $w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]})$ --- 2 oper
- Procedure Warshall-transitive-closure(M_R)

```
W=M_R
for k=1 to n
begin
    for i=1 to n
       begin
          for j=1 to n
          w_ij^[k]= w_ij^[k-1] V (w_ik^[k-1] Λ w_kj^[k-1])
       end
 end
end W is the answer M_R^+
```

<u>Complexity:</u>
$(2n^2)n = O(n^3)$

Equivalence Relation

Definition

- Three important characteristics of the notion "equivalence":
  - Every element is equivalent to itself (reflexivity)
  - If a is equivalent to b, then b is equivalent to a (symmetry)
  - If a is equivalent to b, and b is equivalent to c, then a is equivalent to c (transitivity)
- *A binary relation R on a set A is an equivalence relation if R is reflexive, symmetric and transitive.*

Modular equivalences: Congruence Modulo m

- R={(a,b)| $a \equiv b \pmod{m}$)
- **Reflexive** as aRa
- **Symmetric:**
  - If aRb=>m|(a-b)=>(a-b)=km, where k is an integer
  - Thus, (b-a)=-km=>m|(b-a)=>bRa
- **Transitive:**
  - aRb=>(a-b)=k_1m
  - bRc=>(b-c)=k_2m
  - So, (a-c)=(a-b)+(b-c)=(k_1+k_2)m=>m|(a-c)=>aRc

Equivalence Class

- Let R be an equivalence relation on a set A. The set of all the elements that are related to an element *a* of A is called the equivalence class of *a*. It is denoted by $[a]_R$. When only one relation is under consideration, one can drop the subscript R.
- $[a]_R=\{s|(a,s)\in R\}$. Any element in the class can be chosen as the **representative** element in the class.

Example

- aRb iff a=b or a=-b
- R is an Equivalence relation (exercise)
- What is the equivalence class of an integer a?
- $[a]_R=\{-a,a\}$

Example

- What are the equivalence classes of 0 and 1 for congruence modulo 4?
  - $[0]=\{\ldots,-8,-4,0,4,8,\ldots\}$
  - $[1]=\{\ldots,-7,-3,1,5,9,\ldots\}$
- The equivalence classes are called congruent classes modulo m.

Partitions

- Let R be an equivalence relation on a set A. These statements are equivalent if:

1. aRb
2. [a]=[b]
3. [a]∩[b]≠Ø

- 1=>2=>3=>1

Theorem

- Let R be an equivalence relation on set A.

1. For, all a, b$\in$A, either $[a]=[b]$ or $[a]\cap[b]=\varnothing$
2. $\bigcup_{x\in A}[x]=A$

Thus, the equivalence classes form a partition of A. By partition we mean a collection of disjoint nonempty subsets of A, that have A as their union.

Why both conditions 1 and 2 are required?

- In the class we had a discussion, saying that is 1 sufficient and does 2 always hold?
- Lets consider the following example:

  Define over the set $A=\{y|y \in I^+\}$

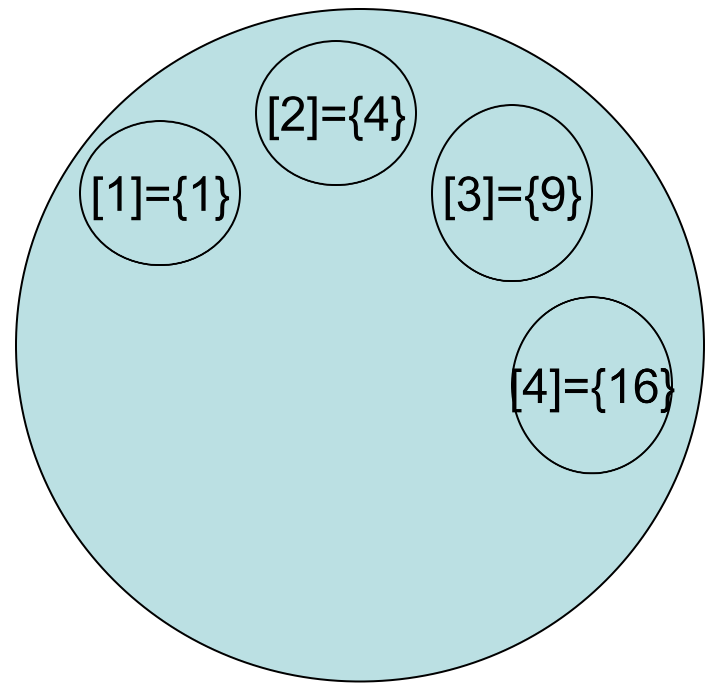
  $$R=\{(a,b)|b=a^2\}.$$

  Thus (1,1),(2,4) are members of R.
- Consider the class: $[x]=\{s|(x,s) \in R\}$

Pictographic Representation

- So, we see that we have classes which satisfy property 1

  (here for distinct a and b, the intersection of [a] and [b] is always null)
- But the union of the partitions is not the set A. It's a subset of A
- For equivalence classes it is exactly A.
- Property 1 and 2 together define equivalence classes.



Quotient Set

- Let R be an equivalence relation on A. The *quotient* set, A/R, is the partition $\{[a]_R | a \in A\}$. The quotient set is also called A modulo R or the partition of A induced by R.
- Equivalence classes of R form a partition of A. Conversely, given a partition $\{A_i | i \in I\}$ of A, there is an equivalence relation R that has the sets, A_i as its equivalence classes.
  - *Equivalence relations induce partitions and partitions induce equivalence relations*