Solving Recurrences

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Recurrence Relations

- A *recurrence relation* (R.R., or just *recurrence*) for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more previous elements
  $a_0, \ldots, a_{n-1}$ of the sequence, for all $n \geq n_0$.
  - *I.e.*, just a recursive definition, without the base cases.
- A particular sequence (described non-recursively) is said to *solve* the given recurrence relation if it is consistent with the definition of the recurrence.
  - A given recurrence relation may have many solutions.

Example

- Consider the recurrence relation

  $a_n = 2a_{n-1} - a_{n-2}$ $(n \geq 2)$.

- Which of the following are solutions?

  $a_n = 3n$

  $a_n = 2^n$

  $a_n = 5$

Further Examples

- Recurrence relation for growth of a bank account with P% interest per given period:

  $$M_n = M_{n-1} + (P/100)M_{n-1}$$

- Growth of a population in which each pair of rabbit yield 1 new one every year after 2 years of their birth.

  $$P_n = P_{n-1} + P_{n-2}$$ (Rabbits and Fibonacci relation)

Solving Compound Interest RR

- $M_n = M_{n-1} + (P/100)M_{n-1}$

  $= (1 + P/100)\, M_{n-1}$

  $= r\, M_{n-1}$ (let $r = 1 + P/100$)

  $= r\,(r\, M_{n-2})$

  $= r \cdot r \cdot (r\, M_{n-3})$ …and so on to…

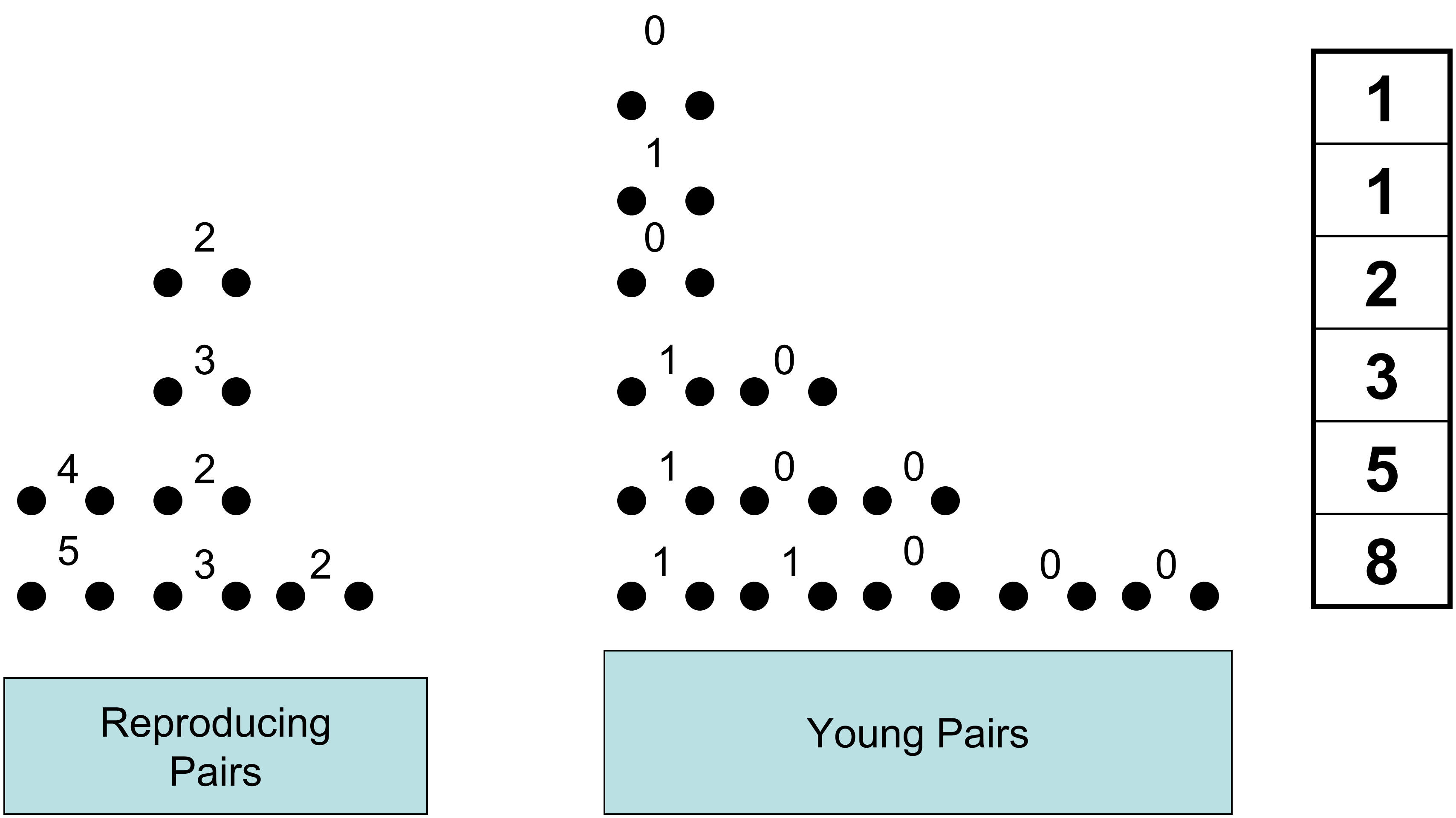
  $= r^n\, M_0$

Rabbits on an Island (assuming rabbits are immortal)

Year	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

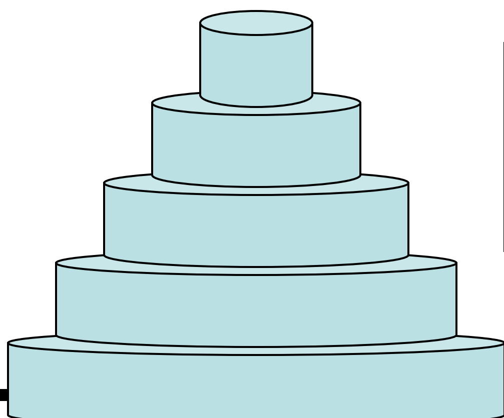
$$P_n = P_{n-1} + P_{n-2}$$

Further Explanation



Tower of Hanoi Example

- Problem: Get all disks from peg 1 to peg 2.
  - Only move 1 disk at a time.
  - Never set a larger disk on a smaller one.

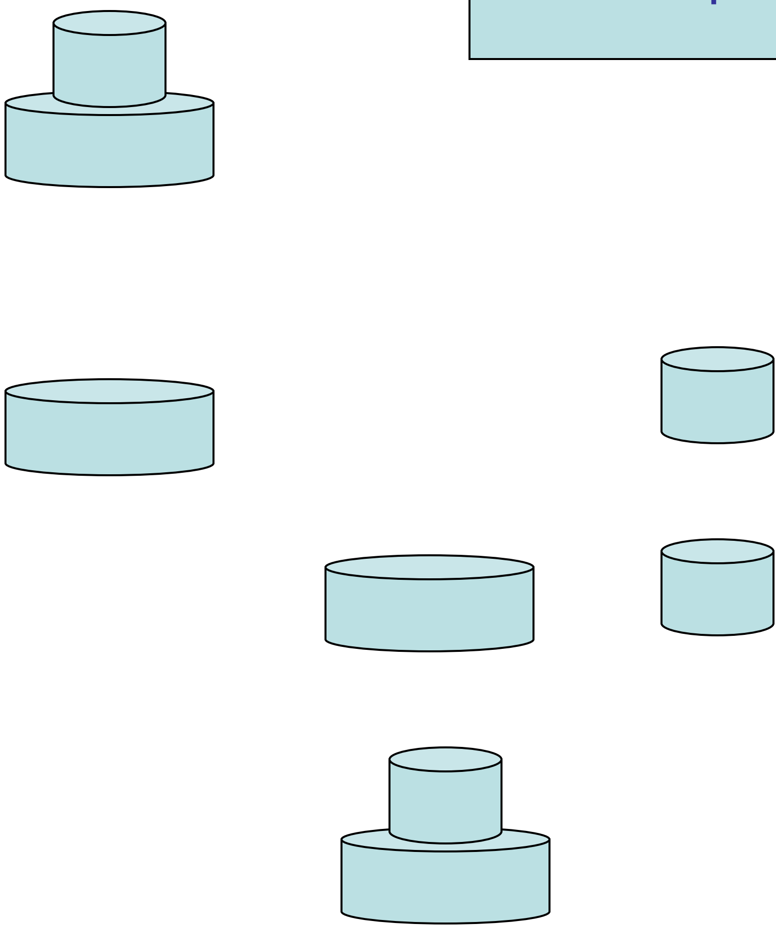


Question: Compute the number of steps H_n

Peg #1	Peg #2	Peg #3

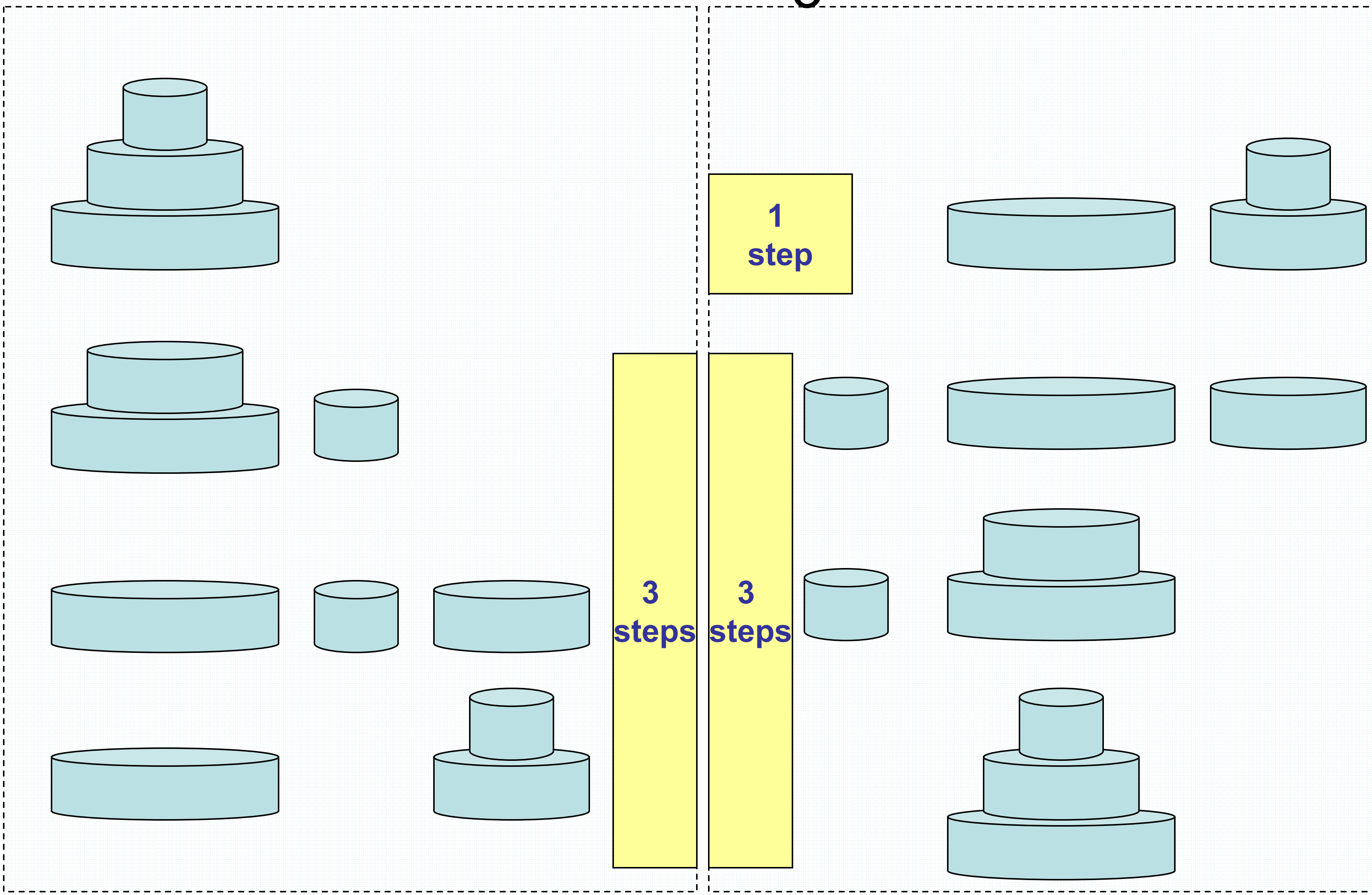
Intuition

H_1=1 is evident



So, H_2=3

For n=3, H_3=7



Hanoi Recurrence Relation

- Let H_n = # moves for a stack of n disks.
- Optimal strategy:
  - Move top $n-1$ disks to spare peg. (H_{n-1} moves)
  - Move bottom disk. (1 move)
  - Move top $n-1$ to bottom disk. (H_{n-1} moves)
- Note: $H_n = 2H_{n-1} + 1$

Solving Tower of Hanoi RR

$$
\begin{aligned}
H_n &= 2\,H_{n-1} + 1 \\
&= 2\,(2\,H_{n-2} + 1) + 1 &&= 2^2\,H_{n-2} + 2 + 1 \\
&= 2^2(2\,H_{n-3} + 1) + 2 + 1 &&= 2^3\,H_{n-3} + 2^2 + 2 + 1 \\
&\dots \\
&= 2^{n-1}\,H_1 + 2^{n-2} + \dots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + \dots + 2 + 1 && \text{(since } H_1 = 1\text{)} \\
&= \sum_{i=0}^{n-1} 2^i \\
&= 2^n - 1
\end{aligned}
$$

Another R.R. Example

- Find a R.R. & initial conditions for the number of bit strings of length *n* without two consecutive 0s. Assume $n \geq 3$.
- We can solve this by breaking down the strings to be counted into cases that end in 0 and in 1.
  - For each ending in 0, the previous bit must be 1, and before that comes any qualifying string of length $n-2$.
  - For each string ending in 1, it starts with a qualifying string of length $n-1$.
- Thus, $a_n = a_{n-1} + a_{n-2}$. (Fibonacci recurrence.)

Yet another R.R. example…

- Give a recurrence (and base cases) for the number of n-digit decimal strings containing an *even* number of 0 digits.
- Can break down into the following cases:
  – Any valid string of length $n-1$ digits, with any digit 1-9 appended.
  – Any invalid string of length $n-1$ digits, + a 0.
- $a_n = 9a_{n-1} + (10^{n-1} - a_{n-1})$
  $= 8a_{n-1} + 10^{n-1}$.
  – Base cases: $a_0 = 1$ (ε), $a_1 = 9$ (1-9).

Catalan Numbers

- Find a R.R for the number of ways we can parenthesize the product of n+1 numbers, x_0, x_1, …, x_n to specify the order of multiplication. Call it C_n.
- **Define $C_0=C_1=1$ (its important to have proper base cases)**
- If n=2, $(x_0.x_1).x_2, x_0.(x_1.x_2) \Rightarrow C_2=2$
  - ***Note that $C_2=C_1C_0+C_0C_1=1+1=2$***
- If n=3, $((x_0.x_1).x_2).x_3$; $(x_0.x_1).(x_2.x_3)$; $(x_0.(x_1.x_2)).x_3$; $x_0.((x_1.x_2).x_3)$; $x_0.(x_1.(x_2.x_3))$

  => $C_3=5$
  - ***Note that $C_3=C_2C_0+ C_1C_1+C_0C_2=2+1+2=5$***

Catalan Numbers

- The final "." operator is outside the scope of any parenthesis.
- The final . can be between any x_k and x_{k+1} out of the n+1 numbers.
- How many ways can we have parenthesis as follows:
  - $[x_0, x_1, \ldots, x_k] . [x_{k+1}, x_{k+2}, \ldots, x_n]$

C_k

C_{n-k-1}

  - The "." can occur in after any x_k, where k ranges from 0 to n-1
  - So, the total number of possible parenthesis is:

$$\sum_{i=0}^{n-1} C_k C_{n-k-1}$$

Exact form of C_n can be computed using Generating functions

Solving Recurrences

- A *linear homogeneous recurrence of degree k with constant coefficients* ("*k*-LiHoReCoCo") is a recurrence of the form

  $$a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k},$$

  where the c_i are all real, and $c_k \neq 0$.

- The solution is uniquely determined if *k* initial conditions $a_0 \ldots a_{k-1}$ are provided. This follows from the second principle of Mathematical Induction.

Examples

- $f_n=f_{n-1}+f_{n-2}$ is a k-LiHoReCoCo
- $h_n=2h_{n-1}+ 1$ is not Homogenous
- $a_n=a_{n-1}+ a_{n-2}^2$ is not linear
- $b_n=nb_{n-1}$ does not have a constant co-efficient

Solving LiHoReCoCos

- The basic idea: Look for solutions of the form $a_n = r^n$, where r is a constant not zero (r=0 is trivial)
- This requires the *characteristic equation*:

  $r^n = c_1 r^{n-1} + \ldots + c_k r^{n-k}$, *i.e.*, (rearrange & × by r^{k-n})

  $r^k - c_1 r^{k-1} - \ldots - c_k = 0$
- The solutions (*characteristic roots*) can yield an explicit formula for the sequence.

Solving 2-LiHoReCoCos

- Consider an arbitrary 2-LiHoReCoCo:

  $a_n = c_1 a_{n-1} + c_2 a_{n-2}$
- It has the characteristic equation (C.E.):

  $r^2 - c_1 r - c_2 = 0$
- **Theorem 1:** If the CE has 2 roots $r_1 \neq r_2$, then $\{a_n\}$ is a solution to the RR iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n \geq 0$ for constants α_1, α_2.

Example

- Solve the recurrence $a_n = a_{n-1} + 2a_{n-2}$ given the initial conditions $a_0 = 2$, $a_1 = 7$.
- Solution: Use theorem 1:
  - $c_1 = 1$, $c_2 = 2$
  - Characteristic equation:
    $r^2 - r - 2 = 0$
  - Solutions: $r = [-(-1) \pm ((-1)^2 - 4 \cdot 1 \cdot (-2))^{1/2}] / 2 \cdot 1$
    $= (1 \pm 9^{1/2})/2 = (1 \pm 3)/2$, so $r = 2$ or $r = -1$.
  - So $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$.

(Using the quadratic formula here.)

$$ax^2 + bx + c = 0 \Leftrightarrow$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example Continued…

- To find α_1 and α_2, solve the equations for the initial conditions a_0 and a_1:

  $$a_0 = 2 = \alpha_1 2^0 + \alpha_2 (-1)^0$$
  $$a_1 = 7 = \alpha_1 2^1 + \alpha_2 (-1)^1$$

  Simplifying, we have the pair of equations:

  $$2 = \alpha_1 + \alpha_2$$
  $$7 = 2\alpha_1 - \alpha_2$$

  which we can solve easily by substitution:

  $$\alpha_2 = 2 - \alpha_1; \quad 7 = 2\alpha_1 - (2 - \alpha_1) = 3\alpha_1 - 2;$$
  $$9 = 3\alpha_1; \quad \alpha_1 = 3; \quad \alpha_2 = 1.$$

- Final answer: $a_n = 3 \cdot 2^n - (-1)^n$

Check: $\{a_{n\geq 0}\} = 2, 7, 11, 25, 47, 97 \ldots$

Proof of Theorem 1

- Proof that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is always a solution:
  - We know $r_1^2 = c_1 r_1 + c_2$ and $r_2^2 = c_1 r_2 + c_2$.
  - Now we can show the proposed sequence satisfies the recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2}$:

$$
\begin{aligned}
c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\
&= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\
&= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 = \alpha_1 r_1^n + \alpha_2 r_2^n = a_n. \quad \square
\end{aligned}
$$

This shows that if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, then $\{a_n\}$ is a solution to the R.R.

The remaining part of the proof

- If $\{a_n\}$ is a solution of R.R. then, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, for n=0,1,2,…
- Can complete proof by showing that for any initial conditions, we can find corresponding α's
  - $a_0 = C_0 = \alpha_1 + \alpha_2$
  - $a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2$
  - $\alpha_1 = (C_1 - C_0 r_2)/(r_1 - r_2)$; $\alpha_2 = (C_0 r_1 - C_1)/(r_1 - r_2)$
  - But it turns out this is a solution only if $r_1 \neq r_2$. *So the roots have to be distinct.*
  - *The recurrence relation and the initial conditions determine the sequence uniquely. It follows that*

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ *(as we have already shown that this is a soln.)*

The Case of Degenerate Roots

- Now, what if the C.E.

  $r^2 - c_1 r - c_2 = 0$ has only 1 root r_0?

- **Theorem 2:** Then,

  $\boldsymbol{a_n = (\alpha_1 + \alpha_2 n) r_0^n}$, for all $n \geq 0$,

  for constants α_1, α_2.

Example

- Solve: $a_n=6a_{n-1}-9a_{n-2}$ with $a_0=1,a_2=6$
- CE is $r^2-6r+9=0 \Rightarrow r_0=3$
- So, the general form of the soln is:
  - $a_n=(\alpha_1 + \alpha_2 n)3^n$
  - Solve the rest using the initial conditions

k-LiHoReCoCos

- Consider a *k*-LiHoReCoCo: $a_n = \sum_{i=1}^{k} c_i a_{n-i}$
- It's C.E. is:

$$r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$$

- **Thm.3:** If this has *k* distinct roots r_i, then the solutions to the recurrence are of the form:

$$a_n = \sum_{i=1}^{k} \alpha_i r_i^n$$

for all $n \geq 0$, where the α_i are constants.

Example

- Solve:

  $a_n=6a_{n-1}-11a_{n-2}+6a_{n-3}$, with initial conditions $a_0=2$, $a_1=5$ and $a_2=15$.

CE is $r^3-6r^2+11r-6=0 \Rightarrow (r-1)(r-2)(r-3)=0$

Thus the soln is:

$$a_n=(\alpha_1 1^n+ \alpha_2 2^n+ \alpha_3 3^n)$$

Solve the rest.

Degenerate *k*-LiHoReCoCos

- Suppose there are t roots $r_1,\ldots,r_t$ with multiplicities $m_1,\ldots,m_t$. Then:

$$a_n = \sum_{i=1}^{t}\left(\sum_{j=0}^{m_i-1}\alpha_{i,j}n^j\right)r_i^n$$

for all $n \geq 0$, where all the α are constants.

Example

- Solve: $a_n=-3a_{n-1}-3a_{n-2}-a_{n-3}$, $a_0=1$, $a_1=-1$, $a_2=-1$
- CE is : $r^2+3r+3r+1=(r+1)^3=0$ => $r=-1$ with multiplicity 3.
- So, soln is :
  - $a_n=(\alpha_1+ \alpha_2 r+ \alpha_3 r^2)(-1)^n$
  - Complete the rest.

Li<u>No</u>ReCoCos

- Linear *<u>no</u>nhomogeneous* RRs with constant coefficients may (unlike Li**<u>Ho</u>**ReCoCos) contain some terms $F(n)$ that depend *only* on n (and *not* on any a_i's). General form:

$$\underbrace{a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k}} + F(n)$$

The *associated homogeneous recurrence relation* (associated Li<u>Ho</u>ReCoCo). **F(n) is not identically zero.**

Solutions of LiNoReCoCos

- A useful theorem about LiNoReCoCos:
  - If $a_n = p(n)$ is any *particular* solution to the LiNoReCoCo

$$a_n = \left(\sum_{i=1}^{k} c_i a_{n-i} \right) + F(n)$$

  - Then *all* its solutions are of the form:

$$a_n = p(n) + h(n),$$

    where $a_n = h(n)$ is any solution to the associated homogeneous RR $a_n = \left(\sum_{i=1}^{k} c_i a_{n-i} \right)$

LiNoReCoCo Example

- Find all solutions to $a_n = 3a_{n-1}+2n$. Which solution has $a_1 = 3$?
  - Notice this is a 1-LiNoReCoCo. Its associated 1-LiHoReCoCo is $a_n = 3a_{n-1}$, whose solutions are all of the form $a_n = \alpha 3^n$. Thus the solutions to the original problem are all of the form
    $a_n = p(n) + \alpha 3^n$. So, all we need to do is find one $p(n)$ that works.

Trial Solutions

- If the extra terms $F(n)$ are a degree-t polynomial in n, you should try a degree-t polynomial as the particular solution $p(n)$.
- This case: $F(n)$ is linear so try $a_n = cn + d$.

  $cn+d = 3(c(n-1)+d) + 2n$ (for all n)

  $(2c+2)n + (2d-3c) = 0$ (collect terms)

  So $c = -1$ and $d = -3/2$.

  So $a_n = -n - 3/2$ is a solution.
- Check: $a_{n\geq 1} = \{-5/2, -7/2, -9/2, \ldots\}$

Finding a Desired Solution

- From the previous, we know that all general solutions to our example are of the form:

  $a_n = -n - 3/2 + \alpha 3^n$.

  Solve this for α for the given case, $a_1 = 3$:

  $3 = -1 - 3/2 + \alpha 3^1$

  $\alpha = 11/6$

- The answer is $a_n = -n - 3/2 + (11/6)3^n$.

Double Check Your Answer!

- Check the base case, $a_1=3$:

$$a_n = -n - 3/2 + (11/6)3^n$$
$$a_1 = -1 - 3/2 + (11/6)3^1$$
$$= -2/2 - 3/2 + 11/2 = -5/2 + 11/2 = 6/2 = 3$$

- Check the recurrence, $a_n = 3a_{n-1}+2n$:

$$-n - 3/2 + (11/6)3^n = 3[-(n-1) - 3/2 + (11/6)3^{n-1}]+2n$$
$$= 3[-n - 1/2 + (11/6)3^{n-1}] + 2n$$
$$= -3n - 3/2 + (11/6)3^n + 2n = -n - 3/2 + (11/6)3^n \blacksquare$$

Theorem

- Suppose that $\{a_n\}$ satisfies the LiNoReCoCo, $a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k}$ + F(n), where c_1, c_2, …, c_n are real numbers and $F(n)=(b_t n^t+b_{t-1} n^{t-1}+\ldots+b_0)s^n$, where b's and s are real numbers.

- When s is not a root of the CE, there is a Particular solution of the form: $(p_t n^t+p_{t-1} n^{t-1}+\ldots+p_0)s^n$.
- When s is a root of this CE and its multiplicity is m, there is a particular solution of the form: $n^m(p_t n^t+p_{t-1} n^{t-1}+\ldots+p_0)s^n$

State the Particular Solutions

- RR: $a_n=6a_{n-1}-9a_{n-2}+F(n)$
- CE has a single root 3, with multiplicity 2.
- $F(n)=3^n$ Particular Solution: $p_0n^2 3^n$
- $F(n)=n3^n$ Particular Solution:
  $n^2 (p_1n+p_0)3^n$
- $F(n)=n^2 2^n$ Particular Solution:
  $(p_2n^2+p_1n+p_0)3^n$
- $F(n)=(n^2+1)3^n$ Particular Solution:
  $n^2(p_2n^2+p_1n+p_0)3^n$

Be Careful when s=1

- Example: $a_n=a_{n-1}+n$, $a_1=1$
- CE: r=1, with multiplicity 1
- F(n)=n, Particular Solution is $n(p_1n+p_0)$
- Solve for p_1 and p_0 using the recurrence equation
- Write the solution: c (solution to the associated homogenous RR) + Particular Solution
- Solve for c using $a_1=1$ and obtain $a_n=n(n+1)/2$